FOR THE
IB DIPLOMA
PROGRAMME

Mathematics

ANALYSIS AND APPROACHES **HL**

EXAM PRACTICE WORKBOOK

Paul Fannon
Vesna Kadelburg
Stephen Ward

Boost

HODDER EDUCATION

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Introduction

Revising for exams can sometimes be overwhelming. This book is designed to help you structure your revision and provide opportunities to practise sitting exam-style papers. Revision should be a cycle of going through your notes and textbooks, practising exam-style questions, reviewing your strengths and weaknesses, and returning to your notes and textbooks.

There are five skills that are needed for exam success:

- knowledge of all the topics required in the syllabus
- facility with basic algebra and arithmetic
- familiarity with your calculator
- the ability to make links and solve problems
- calmness under test conditions.

You need to be aware of your own strengths and weaknesses in each of these areas. This revision guide is designed to help with each skill.

How to use this book

The book comprises four sections that are designed to help you master the five skills listed above.

Calculator checklist

This lists all the tools provided by your GDC (graphical display calculator) that you need to be familiar with. Different calculators might have slightly different input methods, so it is best to use your own calculator manual (these can be found online) to find out the exact syntax yours uses.

Syllabus revision

This section goes through the syllabus line by line to make sure you have covered every part thoroughly. Each skill required in the syllabus is exemplified by a question. You can either start by going over the syllabus content, or by doing the questions. These questions illustrate the basic skills you need; however, they are not all designed to be exam-style questions as they are designed to check syllabus understanding rather than problem-solving. The answers to these questions can be found online at www.hoddereducation.co.uk/ib-extras. Once you are happy, tick the 'revised' box. If you need more details, there are references to the section in the accompanying Hodder Education *Mathematics for the IB Diploma: analysis and approaches SL* or *Mathematics for the IB Diploma: analysis and approaches HL* textbook corresponding to each syllabus item.

Questions with calculator icons are designed specifically to test calculator or non-calculator skills. Those without an icon could be done either with or without a calculator.

Paper plan

This table provides an overview of the entire syllabus that maps the practice papers in this book and in the *Mathematics for the IB Diploma: analysis and approaches HL* textbook to the different topics and also serves as a revision checklist. You should use the mastery section to tick off and make sure that you have covered each topic. When you have revised the topic, you can tick the second column. Then try doing some questions – either from the textbook or the practice papers – and tick the final column once you feel confident with the topic.

The practice paper section shows the corresponding topic for each question in the textbook practice papers and the sets of practice papers in this book. You can use this to check the type of questions that you might get on each topic.

Practice papers and mark schemes

The best way to practise for exams is to do exams. These papers are designed to mimic the style of real IB papers. The questions often combine more than one syllabus topic and can require you to make novel links. As in the real exam papers, there is space for you to write in your calculations and answers to questions in Section A; for Section B, you will need to use a separate notebook.

Understanding mark schemes

Once you have tried some of the practice papers in this book, it is a good idea to mark your own (and also mark other people's) to see what makes things easy or difficult to award marks.

There are three different types of marks awarded in exams:

M These are method marks. They are awarded for a clear and obvious attempt to use the correct method. There is a certain amount of subjective opinion needed to award these. For example, if you are asked to find the length of the hypotenuse, h, of a right-angled triangle with shorter sides 5 and 7, which of the following would be awarded a method mark?

I $h = 5 + 7 = 12$

II $h = \sin(5) + \cos(7) = 1.08$

III $h = \sqrt{5+7} = 3.46$

IV $h = 5^2 + 7^2 = 74$

V $h = \sqrt{7^2 - 5^2} = 4.90$

VI $h = \sqrt{5^2 + 7^2} = 5 + 7 = 12$

Most examiners would agree that the first three examples are not good enough to award a method mark. In case VI, even though there is subsequent incorrect working and the wrong answer, a method mark would still be awarded. Cases IV and V are on the boundary of what might be acceptable and would probably require discussion among the examiners to find a clear boundary, but it is likely both answers would be awarded a method mark. However, an answer of 74 or 4.90 by itself would not be awarded any marks because, even though we might have suspicions about where these numbers have come from, it has not been clearly communicated.

Sometimes method marks have brackets around them, for example, (**M1**). In this case they do not have to be explicitly seen and can be inferred from the correct answer.

Remember that sometimes the question requires a particular method (for example, find the maximum value of the function by differentiating) or it might require you to explicitly use the previous working (generally indicated by using the word 'hence'). If you use a different method in either of these instances, even if it works, you will not normally gain any credit.

For Paper 2, many questions will be answered primarily by using a calculator. However, you can still get some method marks for communicating what you are doing. Remember to write down any numbers that you put into your calculator that are not given in the question (for example, midpoints of grouped data). If you are using a graph to solve an equation, then draw a quick sketch.

A These are accuracy marks. They are for obtaining the correct answer. If there is a previous method mark without a bracket around it, then these marks can only be awarded if the previous method mark was awarded (this tends to happen in situations where examiners think the correct answer can be guessed and so they need to see supporting evidence, or when the question was a 'show that' or 'proof' question, where the communication rather than just the final answer is assessed).

Often lines are denoted **M1A1** – this means one method mark for a correct attempt and one accuracy mark for doing it correctly.

The accuracy mark is withheld if the value or expression given is wrong; however, it can also be withheld if the answer is written in calculator notation (for example, $1.8E9$ rather than 1.8×10^9) or is given to the wrong accuracy – remember that all final answers should be either exact or written to three significant figures unless the question says otherwise. It is usually a good idea to write down intermediate working to more significant figures to ensure that the final answer is correct to at least three significant figures (and ideally store the answer to the full accuracy your calculator can hold using the calculator memory store).

Accuracy marks are also awarded when sketching graphs. It is important to choose an appropriate window to show all the important features of the graph and to label any relevant features (for example, axis intercepts, turning points, asymptotes).

Unless a particular form is required, most equivalent forms are accepted – for example, $x^2 + x$ or $x(x+1)$ would normally both be fine. However, there is also an expectation that you understand the general requirements of the course. For example, if the question asked you to find the area under the curve $y = x^2$ between 0 and 1, the answer $\int_0^1 x^2\,dx$, while *technically* equivalent to the correct answer, is not sufficiently simplified – the acceptable answer would be $\frac{1}{3}$ or 0.333.

R These are marks awarded for clear reasoning – often in response to a question asking for explanation or justification. They might also be used when choosing particular solutions from equations (for example, saying that the solution of a quadratic that is negative cannot be a probability).

You may also see an **AG** notation in the mark schemes. This is used when the answer is given in the question. It is to remind the examiner that the correct answer does not mean that people have reasoned properly and to be particularly watchful for flawed arguments that just happen upon the right answer.

Sometimes later parts of a question need you to use an answer from a previous part. If you got the earlier part of the question wrong, the examiner will try to award 'follow through' marks by checking whether your method works for your prior incorrect answer. However, follow through marks may only be awarded if you have clearly communicated how you are using your previous answer, if you have not fundamentally changed the nature of the question (for example, solving a quadratic equation turned into solving a linear equation) and if your answer is not implausible (for example, a negative probability).

Revision tips

- Do not leave all your revision until the last minute. The IB is a two-year course with many later topics building on previous topics. One psychological study suggested that you need to 'learn' something seven times for it to really be fixed in your mind. Try to use each class test, mock exam or new topic as an opportunity to revise previous work.
- Revision should be an active rather than a passive process – often you can read through notes for hours and gain very little new knowledge. Try to do some questions first, then read through your notes and textbooks when you get stuck. Your reading will be far more focused if you are trying to find the solution to a particular difficulty.
- Try varied memorization strategies until you find one that works for you – copying out pages of notes does not work for most people. Strategies that do work for some people include using colour to prioritize key facts, using mind maps and making up silly songs to memorize techniques. Psychologists have found a strong link between memory and smell, so you could try using a particular perfume or deodorant while revising, then using the same one in the final exam!
- Work positively with others – some group revision can be a great way of improving your understanding as you can bounce ideas off each other, try to explain a concept to someone who is struggling or design exam-style questions for your friends to do. However, do be careful – avoid feeling bad by comparing yourself to people who seem to be good at everything and do not be tempted to feel good about yourself by making others feel bad – neither scenario is productive.
- Practise checking your answers. This is a vital skill that you will not suddenly be able to do in the final exam if you never do it in your revision. Think about good ways to check answers; for example, with and without a calculator, working backwards and sanity checking that the answer is plausible.
- Become an expert at using your exam calculator. You cannot start working on this skill too early, as each calculator has its own quirks that you need to get used to. Make sure you are using it in the mode required for the exam and know what happens when the memory is cleared and it is reset ahead of the exam; for example, does it default to radians or degrees mode?
- Become familiar with the exam formula booklet. It contains lots of useful information, but only if you are used to using it – make sure you know what all the symbols mean and where everything is, well ahead of your final exam. Formulae that are included in the formula booklet are indicated in the syllabus content sections of this book by this icon .
- Make sure some of your revision is under timed conditions. During the exam, the time flashes by for some people whereas others have to pace themselves or they run out of steam towards the end of an exam.
- Do not get downhearted if you are getting lots of things wrong, especially at the beginning of the revision process. This is absolutely normal – in fact, you learn a lot more from the things you get wrong than from the things you get right!

- Weirdly, too much revision can be counterproductive. You will have your own personal concentration span beyond which there is no point in revising without a small break. Check that your revision plan is achievable, and schedule in plenty of relaxation time.
- Try to get into stable sleeping and eating patterns in the run-up to the exam. If you are getting up each day at noon and never having caffeine, then a 9 a.m. exam with lots of coffee is unlikely to go well!
- Unless you know that you only have a very good short-term memory, it is unlikely that the night before an exam is the best time to revise. Going for a run, doing some yoga or reading a good book and having a good night's sleep is likely to be worth far more marks than last minute panic revision.
- If you choose to do any revision between Paper 1 and Paper 2, use the syllabus checklist to check if there are any major topics not covered in the first paper and focus your revision on those.

Exam tips

- Use the reading time wisely. Every mathematics exam starts off with five minutes of reading time in which you are not allowed to write. This time is vital – make sure you read as much of the paper as you can and mentally start making a plan.
- The examiners design the difficulty of the questions to be in increasing order in the short questions, and in increasing order within and between each long question; however, their judgement of difficulty is not always going to align with yours, so do not assume that you should do the questions in order. Many people try all the short questions first, spending too long on the last, often tricky, short question and then either panic or run out of time on the long questions. Sometimes the first long question is the easiest question on the paper, so consider doing that first. There is no substitute for potentially gaining lots of marks early on to build confidence for the rest of the exam.
- Keep checking the time. Each mark equates to approximately one minute – so you do not spend 10 minutes on a question worth only 2 marks. Sometimes you have to abandon one question and move on to the next.
- Do not get dispirited if you cannot do a question – the exam is meant to be challenging and you will not need 100% of the marks to get the grade you are aiming for. The worst thing you can do is let one bad question put you off showing your ability in other questions.
- Look at the mark schemes to understand what is given credit. When many method marks are implied, only putting down the final answer is a high-risk strategy! Even the best mathematicians can make mistakes entering numbers into calculators. Mathematical communication is an important skill so try to convey your reasoning clearly – this has the advantage of enabling you to score some marks even if you make a mistake and of marshalling your ideas so you are more likely to get the answer right in the first place.
- Especially in the long questions, do not assume that just because you cannot do an early part that you cannot do later parts. If you get an early part wrong, follow through marks may still be available in later parts if you clearly communicate the correct method, even if you are using the wrong numbers. Sometimes the later parts of questions do not need the results from earlier parts anyway. The only way that you can guarantee getting no marks for part of a question is by leaving it blank!
- In Paper 2, identify which questions are 'calculator questions'. Too many people try to do these questions using non-calculator techniques that do work, but often use up a lot of time.
- Keeping the exam in perspective is perhaps more important than anything else. While it is of some importance, always remember that exams are artificial and imperfect measurements of ability. How much you can achieve working in silence, under timed conditions and by yourself without any resources on one particular set of questions is not what is most valued in mathematics. It should not be the only outcome from the course that matters, nor should it be how you judge yourself as a mathematician. It is only when you realize this that you will relax and have a chance of showing your true ability.
- Finally, make sure that you understand the command terms used in exams – these are listed below. In particular, 'write down' means that you should be able to do it without any major work – if you find that your answer requires lots of writing, then you have missed something!

Command term	Definition
Calculate	Obtain a numerical answer showing the relevant stages in the working.
Comment	Give a judgment based on a given statement or result of a calculation.
Compare	Give an account of the similarities between two (or more) items or situations, referring to both (all) of them throughout.
Compare and contrast	Give an account of similarities and differences between two (or more) items or situations, referring to both (all) of them throughout.
Construct	Display information in a diagrammatic or logical form.
Contrast	Give an account of the differences between two (or more) items or situations, referring to both (all) of them throughout.
Deduce	Reach a conclusion from the information given.
Demonstrate	Make clear by reasoning or evidence, illustrating with examples or practical application.
Describe	Give a detailed account.
Determine	Obtain the only possible answer.
Differentiate	Obtain the derivative of a function.
Distinguish	Make clear the differences between two or more concepts or items.
Draw	Represent by means of a labelled, accurate diagram or graph, using a pencil. A ruler (straight edge) should be used for straight lines. Diagrams should be drawn to scale. Graphs should have points correctly plotted (if appropriate) and joined in a straight line or smooth curve.
Estimate	Obtain an approximate value.
Explain	Give a detailed account including reasons or causes.
Find	Obtain an answer showing relevant stages in the working.
Hence	Use the preceding work to obtain the required result.
Hence or otherwise	It is suggested that the preceding work is used, but other methods could also receive credit.
Identify	Provide an answer from a number of possibilities.
Integrate	Obtain the integral of a function.
Interpret	Use knowledge and understanding to recognize trends and draw conclusions from given information.
Investigate	Observe, study, or make a detailed and systematic examination, in order to establish facts and reach new conclusions.
Justify	Give valid reasons or evidence to support an answer or conclusion.
Label	Add labels to a diagram.
List	Give a sequence of brief answers with no explanation.
Plot	Mark the position of points on a diagram.
Predict	Give an expected result.
Prove	Use a sequence of logical steps to obtain the required result in a formal way.
Show	Give the steps in a calculation or derivation.
Show that	Obtain the required result (possibly using information given) without the formality of proof. 'Show that' questions do not generally require the use of a calculator.
Sketch	Represent by means of a diagram or graph (labelled as appropriate). The sketch should give a general idea of the required shape or relationship, and should include relevant features.
Solve	Obtain the answer(s) using algebraic and/or numerical and/or graphical methods.
State	Give a specific name, value or other brief answer without explanation or calculation.
Suggest	Propose a solution, hypothesis or other possible answer.
Verify	Provide evidence that validates the result.
Write down	Obtain the answer(s), usually by extracting information. Little or no calculation is required. Working does not need to be shown.

Some advice about Paper 3

Paper 3 is a new type of examination to the IB. It will include long, problem-solving questions. Do not be intimidated by these questions – they look unfamiliar, but they will be structured to guide you through the process. Although each question will look very different, it might help you to think about how these questions are likely to work:

- There might be some 'data collection', which often involves working with your calculator or simple cases to generate some ideas.
- There might be a conjecturing phase where you reflect on your data and suggest a general rule.
- There might be a learning phase where you practise a technique on a specific case.
- There might be a proving phase where you try to form a proof. It is likely that the method for this proof will be related to the techniques suggested earlier in the question.
- There might be an extension phase where you apply the ideas introduced to a slightly different situation.

All these phases have their own challenges, so it is not always the case that questions get harder as you go on (although there might be a trend in that direction). Do not assume that just because you could not do one part you should give up – there might be later parts that you can still do.

Some parts might look very unfamiliar, and it is easy to panic and think that you have just not been taught about something. However, one of the skills being tested is mathematical comprehension so it is possible that a new idea is being introduced. Stay calm, read the information carefully and be confident that you do have the tools required to answer the question, it might just be hidden in a new context.

You are likely to have a lot of data so be very systematic in how you record it. This will help you to spot patterns. Then when you are suggesting general rules, always go back to the specific cases and check that your suggestion always works.

These questions are meant to be interlinked, so if you are stuck on one part try to look back for inspiration. This might be looking at the answers you have found, or it might be trying to reuse a method suggested in an earlier part. Similarly, even more than in other examinations, it is vital in Paper 3 that you read the whole question. Sometimes later parts will clarify how far you need to go in earlier parts, or give you ideas about what types of method are useful in the question.

These questions are meant to model the thinking process of mathematicians. Perhaps the best way to get better at them is to imitate the mathematical process at every opportunity. So the next time you do a question, see if you can spot underlying patterns, generalize them and then prove your conjecture. The more you do this, the better you will become.

Calculator checklist

You should know how to:

	Skill	Got it!	Need to check
General	Change between radian and degrees mode		
	Set output to three significant figures		
	Store answers in calculator memory		
	Edit previous calculations		
Number and algebra	Input and interpret outputs in standard index form ($a \times 10^n$)		
	Use the sequence functions to find terms of an arithmetic and geometric sequence		
	Use tables to display sequences		
	Use the sum function to sum sequences		
	Use the TVM package to answer questions about compound interest and depreciation, including finding unknown interest rates and interest periods		
	Evaluate logarithms to any base		
	Find ${}^nC_r$ and ${}^nP_r$ using technology		
	Do operations with complex numbers in Cartesian form		
	Do operations with complex numbers in polar form		
	Convert between Cartesian and polar form		
	Solve systems of linear equations in up to three unknowns when there is a unique solution		
Functions	Graph equations of the form $y = f(x)$		
	Use the zoom/window functions to explore interesting features of graphs		
	Use the trace function to explore graphs, especially suggesting asymptotes		
	Find axis intercepts of graphs		
	Find the coordinates of local maxima or minima of graphs		
	Find the points of intersection of two graphs		
	Solve quadratic equations		
	Solve equations using solve functions on the calculator		
	Solve polynomial equations		
	Use the modulus function		
Geometry and trigonometry	Solve trigonometric equations graphically		
	Input expressions involving functions such as $\sin^2 x$		
	Input expressions involving the inverse functions, such as arcsin x; it is probably called $\sin^{-1} x$		
	Input expressions involving the reciprocal functions, such as $\sec x$		
	Find the magnitude of a vector		
Statistics and probability	Input data, including from frequency tables and grouped data		
	Find mean, median, mode, quartiles and standard deviation from data		
	Input bivariate data		
	Find Pearson correlation coefficient of data		
	Find equations of y-on-x and x-on-y regression lines		
	Calculate probabilities for a given binomial distribution		
	Calculate probabilities for a given normal distribution		
	Calculate boundary values from probabilities for a given normal distribution		
	Calculate z-scores from probabilities of a normal distribution		
Calculus	Estimate the value of a limit from a table or a graph		
	Find the derivative of a function at a given point		
	Use the calculator to sketch the derivative of a function		
	Find definite integrals		
	Find areas/distances using definite integrals and the modulus (absolute value) function		
	Set up and solve first order differential equations using Euler's method		

Syllabus revision

1 Number and algebra

Syllabus content

S1.1	Standard form		
	Book Section 1B	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Operations with numbers of the form $a \times 10^k$ where $1 \leqslant a < 10$.	Input and interpret numbers of this form on the calculator.	1	☐
	Factorize to add or subtract numbers in standard form.	2	☐
	Use the laws of exponents when multiplying or dividing numbers in standard form.	3	☐

S1.2	Arithmetic sequences and series		
	Book Section 2A	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Use of the formulae for the nth term and the sum of the first n terms of the sequence.	Find the nth term of an arithmetic sequence. Use: $u_n = u_1 + (n-1)d$	4	☐
	Use the formula to determine the number of terms in an arithmetic sequence.	5	☐
	Set up simultaneous equations to find the first term and common difference.	6	☐
	Find the sum of n terms of an arithmetic sequence. There are two formulae in the formula booklet. You should be able to use: $S_n = \frac{n}{2}(2u_1 + (n-1)d)$	7	☐
	Or use: $S_n = \frac{n}{2}(u_1 + u_n)$	8	☐
Use of sigma notation for sums of arithmetic sequences.	Understand how sigma notation relates to arithmetic sequences.	9	☐
	Evaluate expressions using sigma notation.	10	☐
Applications.	Recognize arithmetic sequences from descriptions.	11	☐
	In particular, be aware that simple interest is a type of arithmetic sequence.	12	☐
Analysis, interpretation and prediction where a model is not perfectly arithmetic in real life.	Find the common difference as an average of the differences between terms.	13	☐

S1.3	Geometric sequences and series		
	Book Section 2B	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Use of the formulae for the nth term and the sum of the first n terms of the sequence.	Find the nth term of a geometric sequence. Use: $u_n = u_1 r^{n-1}$	14	☐
	Use the formula to determine the number of terms in a geometric sequence.	15	☐
	Set up simultaneous equations to find the first term and common ratio.	16	☐
	Find the sum of n terms of a geometric sequence using: $S_n = \frac{u_1(r^n - 1)}{r-1} = \frac{u_1(1-r^n)}{1-r}, r \neq 1$	17	☐
Use of sigma notation for sums of geometric sequences.	Understand how sigma notation relates to geometric sequences.	18	☐
	Evaluate expressions using sigma notation.	19	☐
Applications.	Recognize geometric sequences from descriptions.	20	☐

S1.4 Financial applications of geometric sequences

Book Section 2C | Revised ☐

Syllabus wording	You need to be able to:	Question
Financial applications of geometric sequences and series. • Compound interest. • Annual depreciation.	Calculate values of investments with compound interest using financial packages or use: $FV = PV \times \left(1 + \frac{r}{100k}\right)^{kn}$ where: FV is the future value PV is the present value n is the number of years k is the number of compounding periods per year r% is the nominal annual rate of interest.	21 ☐
	Calculate interest rates required for particular outcomes.	22 ☐
	Calculate the number of periods required for a particular outcome.	23 ☐
	Calculate the values of goods suffering from depreciation.	24 ☐
	Calculate the real value of investments after inflation.	25 ☐

S1.5 Exponents and logarithms

Book Section 1A, 1C | Revised ☐

Syllabus wording	You need to be able to:	Question
Laws of exponents with integer exponents.	Evaluate expressions involving integer exponents including using: $a^m \times a^n = a^{m+n}$ $\frac{a^m}{a^n} = a^{m-n}$ $(a^m)^n = a^{mn}$ $a^{-n} = \frac{1}{a^n}$ $(ab)^n = a^n \times b^n$ $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	26 ☐
	Simplify algebraic expressions using the above rules.	27 ☐
Introduction to logarithms with base 10 and e.	Use the fact that $a^x = b$ is equivalent to $\log_a b = x$	28 ☐
	Know that natural logarithms, $\ln x$, are equivalent to $\log_e x$ where e = 2.718…	29 ☐
Numerical evaluation of logarithms using technology.	Use your calculator to evaluate logarithms to base 10 and e.	30 ☐

S1.6 Proof

Book Section 11A | Revised ☐

Syllabus wording	You need to be able to:	Question
Simple deductive proof, numerical and algebraic; how to lay out a LHS to RHS proof.	Write out LHS to RHS proofs.	31 ☐
The symbols and notation for equality and identity.	Know that the $\equiv$ symbol is used to emphasize that a statement is true for all allowed values of a variable.	32 ☐

S1.7	Further exponents and logarithms	
	Book Section 12A, 12B	Revised ☐
Syllabus wording	*You need to be able to:*	*Question*
Laws of exponents with rational exponents.	Work with rational exponents using $a^{\frac{1}{m}} = \sqrt[m]{a}$.	33 ☐
Laws of logarithms.	Work with logarithms to bases other than 10 and e.	34 ☐
	Manipulate logarithms algebraically using the laws: $\log_a xy = \log_a x + \log_a y$, $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$, $\log_a x^m = m \log_a x$	35 ☐
Change of base of a logarithm.	Use $\log_a x = \frac{\log_b x}{\log_b a}$.	36 ☐
Solving exponential equations.	Solve equations where the unknown is in the power by taking logarithms of both sides.	37 ☐

S1.8	Infinite geometric sequences	
	Book Section 13A	Revised ☐
Syllabus wording	*You need to be able to:*	*Question*
Sum of infinite convergent geometric sequences.	Use $S_\infty = \frac{u_1}{1-r}$ to find the sum of an infinite geometric sequence.	38 ☐
	Use the condition $\|r\| < 1$ to check if an infinite geometric sequence is convergent.	39 ☐

S1.9	Binomial expansions	
	Book Section 13B	Revised ☐
Syllabus wording	*You need to be able to:*	*Question*
The binomial theorem: expansion of $(a+b)^n$, $n \in \mathbb{Z}^+$.	Use: $(a+b)^n = a^n + {}^nC_1 a^{n-1}b + \ldots + {}^nC_r a^{n-r}b^r + b^n$	40 ☐
Laws of logarithms.	Approximate calculations using binomial expansions.	41 ☐
	Work with single terms in binomial expansions.	42 ☐
Use of Pascal's triangle and ${}^nC_r$.	Evaluate binomial coefficients using either technology, Pascal's triangle or: ${}^nC_r = \frac{n!}{r!\,(n-r)!}$	43 ☐

H1.10	Counting principles and binomial theorem	
	Book Section 1A, 1B, 2A	Revised ☐
Syllabus wording	*You need to be able to:*	*Question*
Counting principles, including permutations and combinations.	Use the formula $n!$ to find the number of arrangements of n objects in a line.	44 ☐
	Use: ${}^nC_r = \frac{n!}{r!\,(n-r)!}$ to find the number of combinations – arrangements of a subset of r objects from n when order does not matter.	45 ☐
	Use: ${}^nP_r = \frac{n!}{(n-r)!}$ to find the number of permutations – arrangements of a subset of r objects from n when order matters.	46 ☐
	Solve problems involving counting principles.	47 ☐
Extension of the binomial theorem to fractional and negative indices.	Use the binomial theorem for fractional indices: $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 \ldots$	48 ☐
	Use the binomial theorem for negative indices.	49 ☐
	Understand the domain in which binomial theorem expansions are valid.	50 ☐

H1.11	Partial fractions		
	Book Section 2B	Revised ☐	
Syllabus wording	*You need to be able to:*		*Question*
Partial fractions.	Write an expression in terms of partial fractions.		51 ☐

H1.12	Definitions of complex numbers		
	Book Section 4A	Revised ☐	
Syllabus wording	*You need to be able to:*		*Question*
Cartesian form $z = a + b\text{i}$; the terms real part, imaginary part, conjugate, modulus and argument. The complex plane.	Apply the basic rules of arithmetic to complex numbers.		52 ☐
	Find the real and imaginary part of complex numbers, $\text{Re}(z)$ and $\text{Im}(z)$, including when dividing by a complex number.		53 ☐
	Understand the notation z^* and be able to apply it in problems, including comparing real and imaginary parts in equations.		54 ☐
	Understand the terms modulus, $\lvert z\rvert = \sqrt{a^2 + b^2}$ and $\arg z$ where $\tan(\arg z) = \frac{b}{a}$		55 ☐

H1.13	Modulus–argument form		
	Book Section 4B	Revised ☐	
Syllabus wording	*You need to be able to:*		*Question*
Modulus–argument (polar) form: $z = r(\cos\theta + \text{i}\sin\theta) = r\,\text{cis}\,\theta$.	Convert from polar to Cartesian form.		56 ☐
	Convert from Cartesian to polar form.		57 ☐
Euler form: $z = r\text{e}^{\text{i}\theta}$.	Work with numbers in Euler form.		58 ☐
Sums, products and quotients in Cartesian, polar or Euler forms and their geometric interpretations.	Understand that when multiplying two complex numbers the moduli multiply and the arguments add.		59 ☐
	Understand that additions in complex numbers represent vector addition. Understand that multiplication in complex numbers represents rotations and stretches, and use that to perform calculations.		60 ☐

H1.14	De Moivre's theorem		
	Book Section 4C, 4D, 4E	Revised ☐	
Syllabus wording	*You need to be able to:*		*Question*
Complex conjugate roots of quadratic and polynomial equations with real coefficients.	Solve polynomial equations with real coefficients given one root.		61 ☐
De Moivre's theorem and its extension to rational exponents.	Apply De Moivre's theorem: $(\cos\theta + \text{i}\sin\theta)^n = \cos n\theta + \text{i}\sin n\theta$		62 ☐
	Use complex numbers to prove trigonometric identities.		63 ☐
Powers and roots of complex numbers.	Work with numbers in Euler form.		64 ☐

H1.15	Further proof		
	Book Section 5A, 5B, 5C	Revised ☐	
Syllabus wording	*You need to be able to:*		*Question*
Proof by mathematical induction.	Prove sums of sequences using induction.		65 ☐
	Prove divisibility using induction.		66 ☐
	Apply induction to other areas such as complex numbers or differentiation.		67 ☐
Proof by contradiction.	Prove theorems using contradiction.		68 ☐
Use of a counterexample to show that a statement is not always true.	Find and describe counterexamples.		69 ☐

H1.16	**Systems of linear equations**	
Book Section 2C	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*
Solution of systems of linear equations (a maximum of three equations in three unknowns) including cases where this is a unique solution, an infinite number of solutions or no solutions.	Use technology or algebraic methods to solve systems of linear equations.	70 ☐
	Demonstrate that a system has no solution.	71 ☐
	Demonstrate that a system has infinite solutions and be able to describe them using a general solution.	72 ☐

Practice questions

1 Evaluate $(3 \times 10^{40})^2 - 5 \times 10^{80}$.

2 Evaluate $3 \times 10^{97} - 4 \times 10^{96}$, leaving your answer in the form $a \times 10^k$ where $1 \leqslant a < 10$.

3 Evaluate $(6 \times 10^{30}) \div (8 \times 10^{-12})$, leaving your answer in the form $a \times 10^k$ where $1 \leqslant a < 10$.

4 Find the 25th term of the following arithmetic sequence:
20, 17, 14, 11, …

5 An arithmetic sequence has first term 21 and last term 1602. If the common difference is 17, how many terms are in the sequence?

6 An arithmetic sequence has 4th term 10 and 10th term 34. Find the 20th term.

7 Find the sum of the first 30 terms of the arithmetic sequence 13, 10, 7, 4, …

8 An arithmetic sequence has $u_1 = 4$, $u_{20} = 130$. Find the sum of the first 20 terms.

9 Determine the first term and common difference of an arithmetic sequence where the sum of the first n terms is given by $S_n = \sum_{1}^{n} (5r + 11)$.

10 Evaluate $\sum_{1}^{100} (5r + 11)$.

11 On the first day of training for a race, Ahmed runs 500 m. On each subsequent day Ahmed runs 100 m further than the day before. How far has he run in total by the end of the 28th day?

12 Juanita invests \$300 at 2.4% simple interest. How much will be in her account after 10 years?

13 A ball is dropped and the velocity ($v\,\mathrm{m\,s^{-1}}$) measured at different times (t seconds).

t	0	0.1	0.2	0.3
v	0	1.1	1.9	2.7

It is assumed that the velocity when $t = 0$ is correct, but there is uncertainty in the remaining measurements.

a By modelling the situation as an arithmetic sequence, estimate the velocity when $t = 0.5$.

b Make one criticism of the model.

14 Find the 10th term of the geometric sequence 32, −16, 8, −4, …

15 Find the number of terms in the geometric sequence 1, 2, 4, …, 4096.

16 A geometric series has third term 16 and seventh term 256. Find the possible values of the first term and the common ratio.

17 Find the sum of the first eight terms of the geometric sequence 162, 54, 18, …

18 Determine the first term and common ratio of a geometric sequence where the sum of the first n terms is given by $S_n = \sum_{1}^{n} 2 \times 5^r$.

19 Evaluate $S_n = \sum_{1}^{10} 2 \times 5^r$.

20 The population of bacteria in a petri dish grows by 20% each day. There are initially 50 000 bacteria in the dish.

a Find the number of bacteria in the dish after 12 days.

b Explain why this model cannot continue indefinitely.

21 £2000 is invested in an account paying 4% nominal annual interest, compounded monthly. Find the amount in the account after 10 years, giving your answer to two decimal places.

22 Samira wants to invest £1000 in an account paying a nominal annual interest rate of *i*%, compounded quarterly. She wants to have £1500 in her account after 8 years. What value of *i* is required?

23 James invests $100 in an account paying 2.1% annual interest. How many complete years are required until he has doubled his investment?

24 A car suffers from 12% annual depreciation. If the initial value is $40 000, find the value after 4 years.

25 Clint invests $2000 at 3.2% annual compound interest. He estimates that the annual inflation rate is 2.4%. Find the real value of his investment after 5 years, giving your answer to the nearest dollar.

26 Evaluate $(2^{-2})^{-2}$.

27 Simplify $(2x)^3$.

28 Solve $4 \times 10^x = 5$, giving your answer in the form $x = \log_{10} a$.

29 If $e^{2x-6} = 5$, find x in terms of natural logarithms.

30 Evaluate $\ln 10 + \log_{10} e$.

31 Show that $\frac{1}{m-1} + \frac{1}{m+1} \equiv \frac{2}{m^2-1}$.

32 a If $x = ax + b$, find x in terms of a and b.

b If $x \equiv ax + b$ for all x, find the values of a and b.

33 Evaluate $8^{\frac{2}{3}}$.

34 Evaluate $\log_4 32$.

35 If $x = \log_2 10$ and $y = \log_2 5$, write x in terms of y.

36 Write $\log_5 12$ in terms of natural logarithms.

37 Solve $5^{x-1} = 4 \times 3^{2x}$, giving your answer in the form $\frac{\ln a}{\ln b}$ where a and b are rational numbers.

38 Find the sum to infinity of the geometric sequence $2, \frac{2}{3}, \frac{2}{9} \ldots$

39 Find the values of x for which the following geometric series is convergent: $1, 2x, 4x^2 \ldots$

40 Find the binomial expansion of $(2 + x)^4$.

41 Use your answer to question **40** to approximate 2.01^4, giving your answer to four significant figures.

42 Find the constant term in the expansion of $\left(2x - \frac{1}{x^3}\right)^4$.

43 Evaluate ${}^{8}C_3$.

44 How many different ways are there to arrange the letters in the word 'COUNTER'?

45 How many ways are there to pick a committee of 4 from 10 students?

46 How many ways are there to make a three digit number using the digits 1, 2, 3, 4 and 5, with no digits repeated?

47 How many even four digit numbers can be formed using the digits 1, 2, 3, 4 and 5 if each digit can be used at most once?

48 Find the first three terms of the binomial expansion of $\sqrt{1-2x}$, valid for $|x| < \frac{1}{2}$.

49 Find the first three terms of the binomial expansion of $\frac{1}{2+x}$.

50 Find the values of x for which the binomial expansion of $\frac{1}{2+x}$ is valid.

51 Write $\frac{x+4}{(x-2)(x+1)}$ in partial fractions.

52 If $z = 2 + i$ and $w = 1 - 2i$ evaluate $2z + wz$.

53 If a and b are real find the real part of $\frac{1}{a+bi}$.

54 Solve $z + 2z^* = 4 + 6\text{i}$.

55 Find the modulus and argument of $z = 1 - \sqrt{3}\text{i}$.

56 Write $z = 2 + 2\text{i}$ in polar form.

57 Write $z = 4\,\text{cis}\left(\frac{\pi}{6}\right)$ in Cartesian form.

58 Find $\text{Re}\left(\text{e}^{\frac{\text{i}\pi}{3}} + 2\text{e}^{\frac{\text{i}\pi}{6}}\right)$.

59 Evaluate $2\,\text{cis}\left(\frac{\pi}{3}\right) \times 3\,\text{cis}\left(\frac{2\pi}{3}\right)$.

60 Find the complex number obtained by rotating $2 + \text{i}$ through an angle $\frac{\pi}{3}$ anticlockwise around the origin.

61 The cubic equation $x^3 - 4x^2 + 6x - 4 = 0$ has a root $x = 1 + \text{i}$. Find the other two roots.

62 Evaluate z^4 if $z = 2\,\text{cis}\left(\frac{\pi}{4}\right)$.

63 Use De Moivre's theorem to prove that $\cos 3x \equiv 4\cos^3 x - 3\cos x$.

64 Solve $z^3 = 8\mathrm{i}$.

65 Prove that $\sum_{r=1}^{n} 2r - 1 = n^2$.

66 Prove that $7^n - 2^n$ is divisible by 5 for all positive n.

67 Prove De Moivre's theorem, $(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$, for every positive integer n.

68 Prove that the cube root of 5 is irrational.

69 Use a counterexample to demonstrate that $n^2 + 41n + 41$ is not always prime for positive integer n.

70 Use technology to find the solution to the equations:

$$\begin{aligned} x + y + z &= 6 \\ x + y - z &= 0 \\ x + 2y + 3z &= 14 \end{aligned}$$

71 Show that the system

$$\begin{aligned} x + y + z &= 0 \\ x + 3y + 5z &= 2 \\ x + 2y + 3z &= 8 \end{aligned}$$

is inconsistent.

72 Show that the system of equations shown below has infinitely many solutions, and find the general solution.

$$\begin{aligned} x + y - z &= 2 \quad [1] \\ x - 2y + z &= 5 \quad [2] \\ 3x - 3y + z &= 12 \quad [3] \end{aligned}$$

2 Functions

Syllabus content

S2.1	Equation of a straight line		
	Book Section 4A	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Different forms of the equation of a straight line.	Use the gradient-intercept form $y = mx + c$, the general form $ax + by + d = 0$ and the point-gradient form $y - y_1 = m(x - x_1)$ to find the equation of a straight line.	1	☐
	Find the equation of a line given its gradient and a point on the line.	2	☐
	Find the equation of a line given two points on the line. Use: $m = \frac{y_2 - y_1}{x_2 - x_2}$ for the gradient.	3	☐
Parallel lines $m_1 = m_2$.	Find the equation of a line through a given point parallel to another line.	4	☐
Perpendicular lines $m_1 \times m_2 = -1$.	Find the equation of a line through a given point perpendicular to another line.	5	☐

S2.2	Concept of a function		
	Book Section 3A	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Function notation.	Use function notation.	6	☐
Domain, range and graph.	Find the domain of a function.	7	☐
	Find the range of a function.	8	☐
Informal concept that an inverse function reverses or undoes the effect of a function.	Understand that an inverse function reverses the effect of a function.	9	☐
Inverse function as a reflection in the line $y = x$, and the notation $f^{-1}(x)$.	Sketch the graph of the inverse function from the graph of the function.	10	☐

S2.3	Graph of a function		
	Book Section 3B	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Creating a sketch from information given or a context.	Sketch a graph from a list of features or from a given context.	11	☐
Using technology to graph functions.	Sketch the graph of a function from a plot on the GDC.	12	☐

S2.4	Key features of graphs		
	Book Section 3B	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Determine key features of graphs.	Use your GDC to find vertices (maximum and minimum values) and lines of symmetry.	13	☐
	Use your GDC to find vertical and horizontal asymptotes.	14	☐
	Use your GDC to find zeros of functions or roots of equations.	15	☐
Finding the point of intersection of two curves or lines using technology.	Use your GDC to find intersections of graphs.	16	☐

S2.5a	Composite functions		
	Book Section 14A	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Composite functions $(f \circ g)(x) = f(g(x))$.	Find the composite function of two functions.	17	☐
	Find the domain of a composite function.	18	☐

S2.5b	Inverse functions		
	Book Section 14B	Revised ☐	
Syllabus wording	*You need to be able to:*		*Question*
Finding the inverse function $f^{-1}(x)$.	Find the inverse of a function.		19 ☐
	Understand that a function has to be one-to-one to have an inverse.		20 ☐

S2.6	Quadratic functions		
	Book Section 15A	Revised ☐	
Syllabus wording	*You need to be able to:*		*Question*
The quadratic function $f(x) = ax^2 + bx + c$: its graph, y-intercept $(0, c)$.	Identify the shape and y-intercept of the graph of a quadratic function from its equation. Know that if $f(x) = ax^2 + bx + c$ then the axis of symmetry is $x = -\frac{b}{2a}$		21 ☐
The form $f(x) = a(x - p)(x - q)$: x-intercepts $(p, 0)$ and $(q, 0)$.	Identify the x-intercepts of the graph of a quadratic function by factorizing.		22 ☐
The form $f(x) = a(x - h)^2 + k$: vertex (h, k).	Identify the vertex and the line of symmetry of the graph of a quadratic function by completing the square.		23 ☐

S2.7a	Quadratic equations and inequalities		
	Book Section 15B	Revised ☐	
Syllabus wording	*You need to be able to:*		*Question*
Solution of quadratic equations and inequalities.	Solve quadratic equations by factorizing.		24 ☐
	Solve quadratic equations by completing the square.		25 ☐
The quadratic formula.	Solve quadratic equations by using the quadratic formula: $ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$		26 ☐
	Solve quadratic inequalities.		27 ☐

S2.7b	Quadratic discriminant		
	Book Section 15C	Revised ☐	
Syllabus wording	*You need to be able to:*		*Question*
The discriminant $\Delta = b^2 - 4ac$ and the nature of the roots.	Use the discriminant $\Delta = b^2 - 4ac$ to determine whether a quadratic equation has two distinct real roots, one real root or no real roots.		28 ☐
	Use the discriminant to find the set of values of a parameter for which a quadratic equation has a given number of real roots.		29 ☐

S2.8	Rational functions		
	Book Section 16B	Revised ☐	
Syllabus wording	*You need to be able to:*		*Question*
The reciprocal function $f(x) = \frac{1}{x}$, $x \neq 0$: its graph and self-inverse nature.	Sketch the graph of the reciprocal function $y = \frac{1}{x}$ and give the equations of the horizontal and vertical asymptotes.		30 ☐
Rational functions of the form $f(x) = \frac{ax + b}{cx + d}$ and their graphs.	Sketch the graph of rational functions of the form $y = \frac{ax + b}{cx + d}$ and give the equations of the horizontal and vertical asymptotes.		31 ☐

S2.9	Exponential and logarithmic functions	
	Book Section 16C	Revised ☐
Syllabus wording	*You need to be able to:*	*Question*
Exponential functions and their graphs: $f(x) = a^x$, $a > 0$; $f(x) = e^x$.	Sketch the graph of exponential functions.	32 ☐
Logarithmic functions and their graphs: $f(x) = \log_a x$, $x > 0$; $f(x) = \ln x$, $x > 0$.	Sketch the graph of logarithmic functions.	33 ☐
	Understand the relationship between the graphs $y = a^x$ and $y = \log_a x$.	34 ☐
	Change an exponential from base a to base e using $a^x = e^{x \ln a}$.	35 ☐

S2.10a	Solving equations analytically	
	Book Section 17A	Revised ☐
Syllabus wording	*You need to be able to:*	*Question*
Solving equations analytically.	Solve equations by factorizing.	36 ☐
	Solve disguised quadratics.	37 ☐
	Solve equations that lead to quadratics.	38 ☐

S2.10b	Solving equations graphically	
	Book Section 17B	Revised ☐
Syllabus wording	*You need to be able to:*	*Question*
Use of technology to solve a variety of equations, including those where there is no appropriate analytic approach.	Use the graphing feature on your GDC to solve equations of the form $f(x) = 0$.	39 ☐
	Use the graphing feature on your GDC to solve equations of the form $f(x) = g(x)$.	40 ☐

S2.11	Transformations of graphs	
	Book Section 16A	Revised ☐
Syllabus wording	*You need to be able to:*	*Question*
Translations: $y = f(x) + b$; $y = f(x - a)$.	Recognize that $y = f(x) + b$ is a vertical translation by b of $y = f(x)$.	41 ☐
	Recognize that $y = f(x - a)$ is a horizontal translation by a of $y = f(x)$.	42 ☐
Vertical stretch with scale factor p: $y = pf(x)$.	Recognize that $y = pf(x)$ is a vertical stretch with scale factor p of $y = f(x)$.	43 ☐
Horizontal stretch with scale factor $\frac{1}{q}$: $y = f(qx)$.	Recognize that $y = f(qx)$ is a horizontal stretch with scale factor $\frac{1}{q}$ of $y = f(x)$.	44 ☐
Reflections (in both axes): $y = -f(x)$; $y = f(-x)$.	Recognize that $y = -f(x)$ is a reflection in the x-axis of $y = f(x)$.	45 ☐
	Recognize that $y = f(-x)$ is a reflection in the y-axis of $y = f(x)$.	46 ☐
Composite transformations.	Apply two vertical transformations to a graph.	47 ☐
	Apply one horizontal and one vertical transformation to a graph.	48 ☐

H2.12a	Graphs and equations of polynomials	
	Book Section 6A	Revised ☐
Syllabus wording	*You need to be able to:*	*Question*
Polynomial functions, their graphs and equations.	Recognize the shapes of graphs of quadratic, cubic and quartic polynomials.	49 ☐
Zeros, roots and factors.	Use factorization to establish the nature of the zeros of a polynomial and thereby sketch its graph.	50 ☐
	Find the equation of a polynomial from its graph.	51 ☐

H2.12b	The factor and remainder theorems	
	Book Section 6B	Revised ☐
Syllabus wording	*You need to be able to:*	*Question*
The factor and remainder theorems.	Use the remainder theorem.	52 ☐
	Use the factor theorem.	53 ☐
	Factorize cubic and quartic polynomials using the factor theorem.	54 ☐

H2.12c

Sum and product of roots		
Book Section 6C	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*
Sum and product of the roots of polynomial equations.	Find the sum and product of the roots of a polynomial using: Sum $= -\dfrac{a_{n-1}}{a_n}$ Product $= \dfrac{(-1)^n a_0}{a_n}$	55 ☐
	Find a polynomial whose roots are a function of the roots of the original polynomial.	56 ☐
	Find a polynomial with given roots.	57 ☐

H2.13

More rational functions		
Book Section 7A	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*
Rational functions of the form $f(x) = \dfrac{ax+b}{cx^2+dx+e}$ and $f(x) = \dfrac{ax^2+bx+c}{dx+e}$.	Sketch the graph of rational functions of the form $y = \dfrac{ax+b}{cx^2+dx+e}$ and give the equations of the horizontal and any vertical asymptotes.	58 ☐
	Sketch the graph of rational functions of the form $y = \dfrac{ax^2+bx+c}{dx+e}$ and give the equations of the vertical and any oblique asymptotes.	59 ☐

H2.14

Properties of functions		
Book Section 7E	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*
Odd and even functions.	Determine algebraically whether a function is odd, even or neither.	60 ☐
	Determine graphically whether a function is odd, even or neither.	61 ☐
Finding the inverse function $f^{-1}(x)$, including domain restriction.	Find the largest possible domain for which the inverse function exists.	62 ☐
Self-inverse functions.	Determine algebraically whether a function is self-inverse.	63 ☐
	Determine graphically whether a function is self-inverse.	64 ☐

H2.15

Inequalities		
Book Section 7B	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*
Solutions of $g(x) \geqslant f(x)$, both graphically and analytically.	Solve cubic inequalities without technology.	65 ☐
	Solve inequalities graphically with technology.	66 ☐

H2.16a

The modulus function		
Book Section 7C	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*
The graphs of the functions $y = \|f(x)\|$ and $y = f(\|x\|)$.	Sketch graphs of the form $y = \|f(x)\|$.	67 ☐
	Sketch graphs of the form $y = f(\|x\|)$.	68 ☐
Solutions of modulus equations and inequalities.	Solve equations involving the modulus function.	69 ☐
	Solve inequalities involving the modulus function.	70 ☐

H2.16b

More transformations of graphs		
Book Section 7D	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*
The graphs of the functions $y = \dfrac{1}{f(x)}$, $y = f(ax+b)$, $y = [f(x)]^2$.	Sketch graphs of the form $y = \dfrac{1}{f(x)}$.	71 ☐
	Sketch graphs of the form $y = f(ax+b)$.	72 ☐
	Sketch graphs of the form $y = [f(x)]^2$.	73 ☐

Practice questions

1 Find the gradient and y-intercept of the line $3x - 4y - 5 = 0$.

2 Find the equation of the line with gradient –3 passing through the point (2, –4). Give your answer in the form $y = mx + c$.

3 Find the equation of the line passing through the points (–3, –5) and (9, 1). Give your answer in the form $ax + by + d = 0$, where a, b, d are integers.

4 Find the equation of the line through the point (1, 4) parallel to the line $y = 2x - 7$.

5 Find the equation of the line through the point (−2, 3) perpendicular to the line $y = -\frac{1}{4}x + 1$.

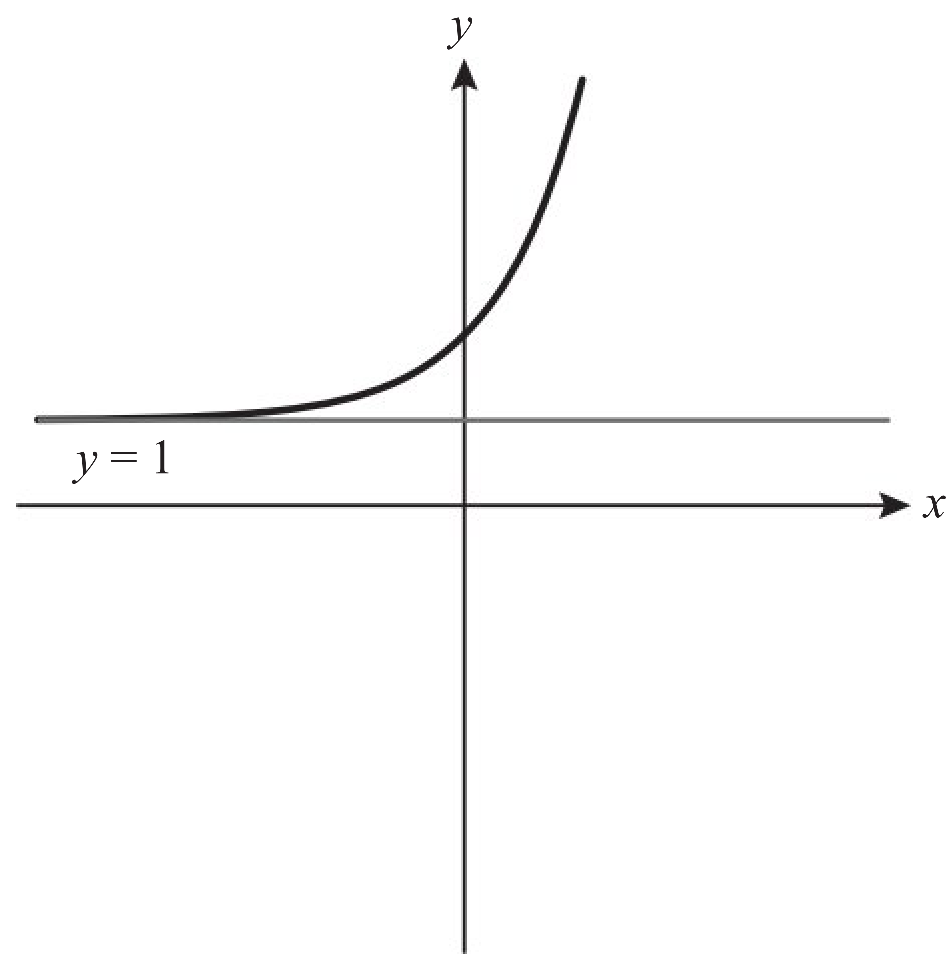
6 If $f(x) = 3x^2 - 4$, find $f(-2)$.

7 Find the largest possible domain of the function $f(x) = \ln(2x - 1)$.

8 Find the range of the function $f(x) = \sqrt{(5 - x)}$, $x \leqslant 1$.

9 If $f(x) = 4 - 3x$, find $f^{-1}(-8)$.

10 Sketch the inverse function of the following graph:



11 The graph of $y = f(x)$ has zeros at -1 and 3 and no vertices. It has a vertical asymptote at $x = 1$ and a horizontal asymptote at $y = -2$.
The range of f is $f(x) > -2$.
Sketch a graph with these properties.

12 Sketch the graph of $y = x^5 - x^4 + 6x^2 - 2$, labelling the y-intercept.

13 a Find the coordinates of the vertices of $y = x^4 + 4x^3 - 3x^2 - 14x - 8$.

b Given that the curve has a line of symmetry, find its equation.

14 Find the equation of all vertical and horizontal asymptotes of the function $f(x) = \dfrac{x^2}{x^2 + x - 6}$.

15 Find the zeros of the function $f(x) = \frac{3}{\sqrt{x}} + 2x - 6$.

16 Find the points of intersection of $y = 3^x$ and $y = 3x + 2$.

17 $f(x) = \frac{1}{x - 2}$ and $g(x) = 3x - 4$.

Find

a $f(g(x))$

b $g(f(x))$.

18 $f(x) = \sqrt{2 - x}$, $x \leqslant 2$ and $g(x) = x - 3$, $x \in \mathbb{R}$.
Find the largest possible domain of $f(g(x))$.

19 $f(x) = \frac{x - 1}{x + 2}$

Find the inverse function $f^{-1}(x)$.

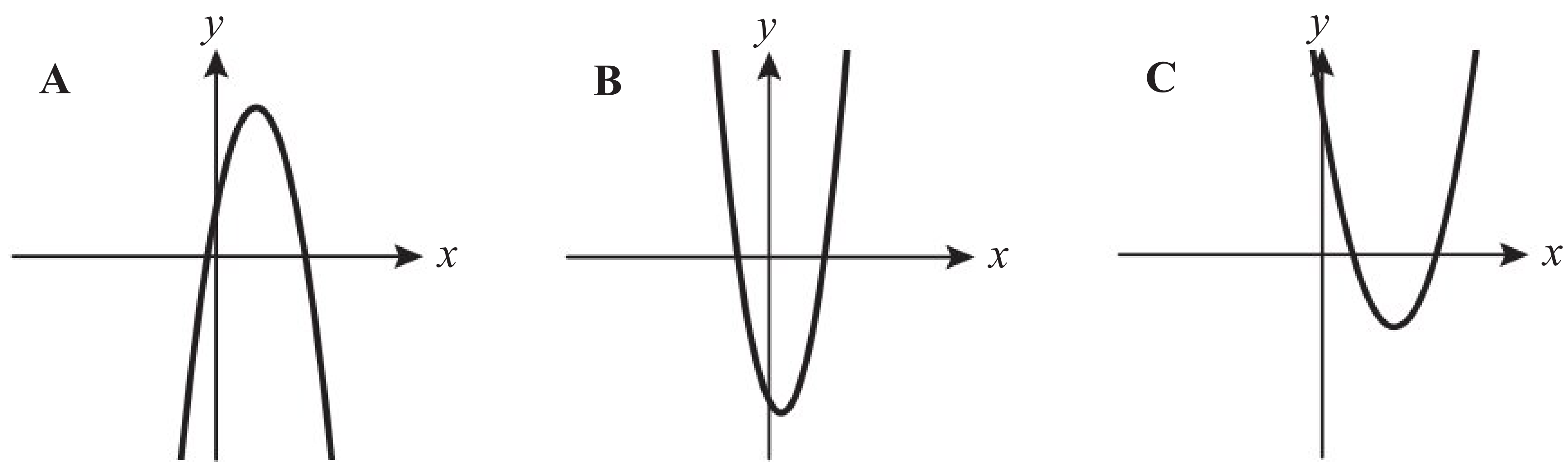
20 Find the largest possible domain of the function $f(x) = 2xe^x$, $x \geqslant k$ for which the inverse f^{-1} exists.

21 Match each equation to the corresponding graph.

a $y = 2x^2 - 6x + 3$

b $y = -3x^2 + 5x + 1$

c $y = 4x^2 - 2x - 3$



22 Sketch the graph of $y = -3x^2 + 15x - 12$, labelling any axis intercepts.

23 a Express $x^2 - 4x + 7$ in the form $(x - h)^2 + k$.

b Hence sketch the graph of $y = x^2 - 4x + 7$, labelling the coordinates of the vertex and any axis intercepts.

24 Solve, by factorizing, the equation $2x^2 + 7x - 15 = 0$.

25 **a** Express $x^2 - 5x + 3$ in the form $(x - h)^2 - k$.

b Hence solve the equation $x^2 - 5x + 3 = 0$.

26 Solve the equation $3x^2 - 4x - 2 = 0$.

27 Solve the inequality $x^2 - x + 12 > 0$.

28 By evaluating the discriminant, determine the number of distinct real roots of the equation $4x^2 + 5x + 3 = 0$.

29 Find the set of values of k for which the equation $3kx^2 + 4x + 12k = 0$ has two distinct real roots.

30 Sketch the graph of $y = \frac{1}{x}$, stating the equations of all asymptotes.

31 Sketch the graph of $y = \dfrac{3x - 2}{x + 1}$, labelling all axis intercepts and asymptotes.

32 Sketch the graph of $y = e^x$, labelling any axis intercepts.

33 Sketch the graph of $y = \log_2 x$, labelling any axis intercepts.

34 On the same axes, sketch the graphs of $y = 0.5^x$ and $y = \log_{0.5} x$, clearly showing the relationship between them.

35 Write 2.8^x in the form e^{kx}, giving the value of k to three significant figures.

36 Solve the equation $x \ln x = 4x$.

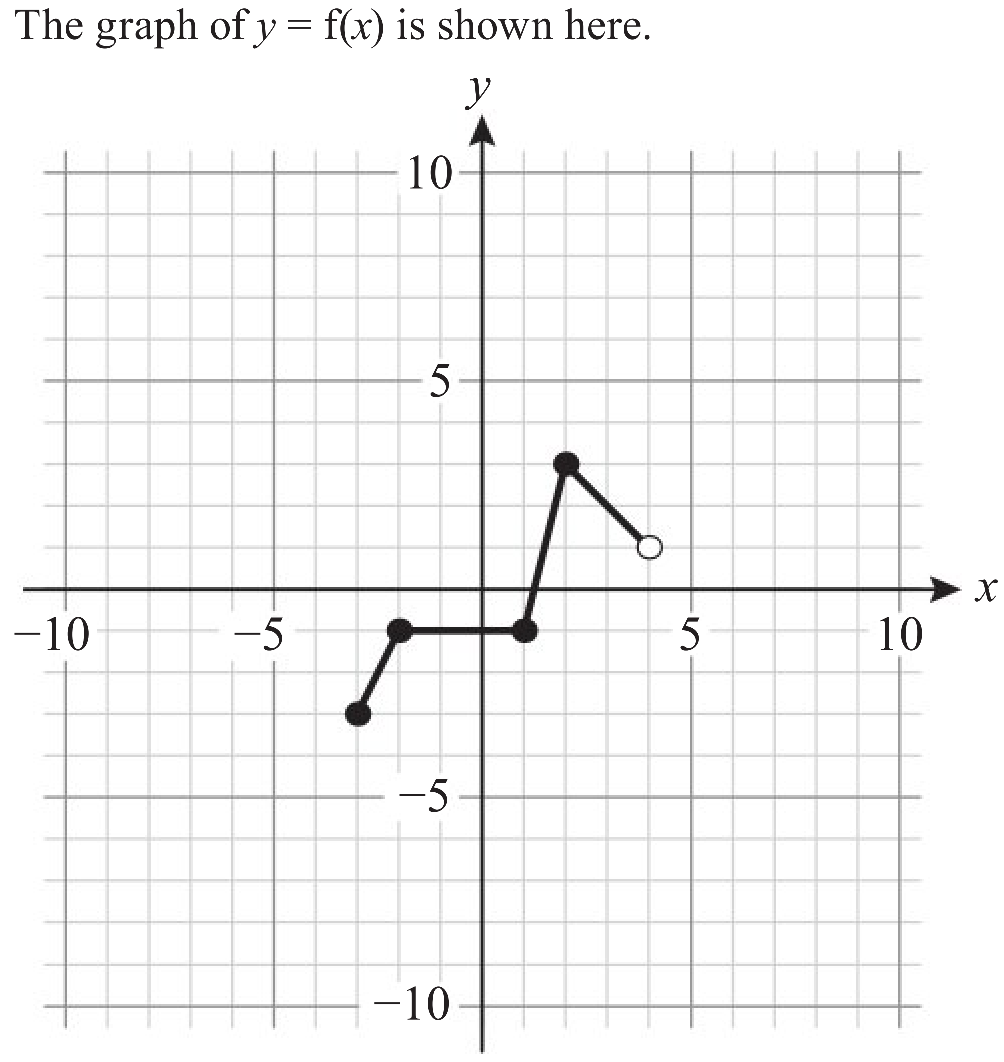
37 Solve the equation $x - 7\sqrt{x} + 10 = 0$.

38 Solve the equation $\log_2 x + \log_2(x + 2) = 3$.

39 Solve the equation $x^2e^x - 2x - 3 = 0$.

40 Solve the equation $2\sin x = x^3 - x + 1$.

41 The graph of $y = f(x)$ is shown here.

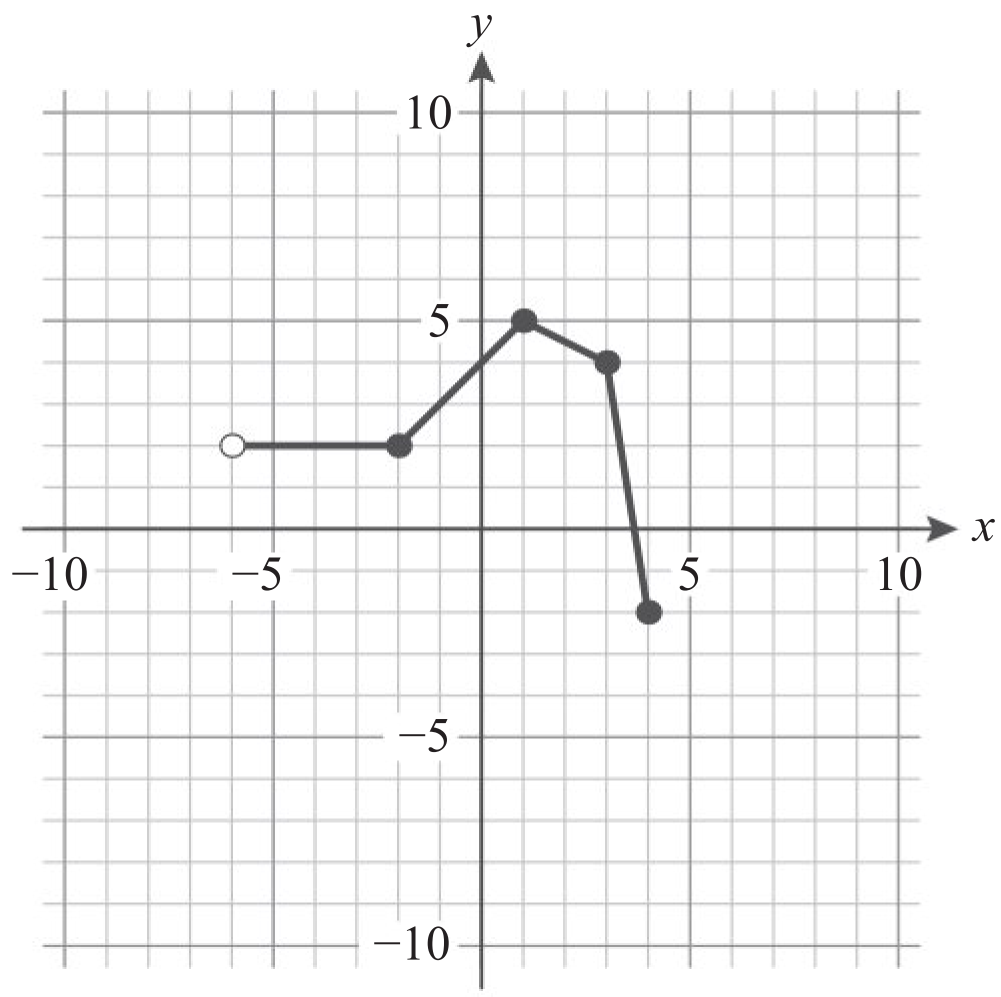


Sketch the graph of $y = f(x) - 4$.

42 The graph of $y = x^2 - 2x + 5$ is translated 3 units to the right. Find the equation of the resulting graph in the form $y = ax^2 + bx + c$.

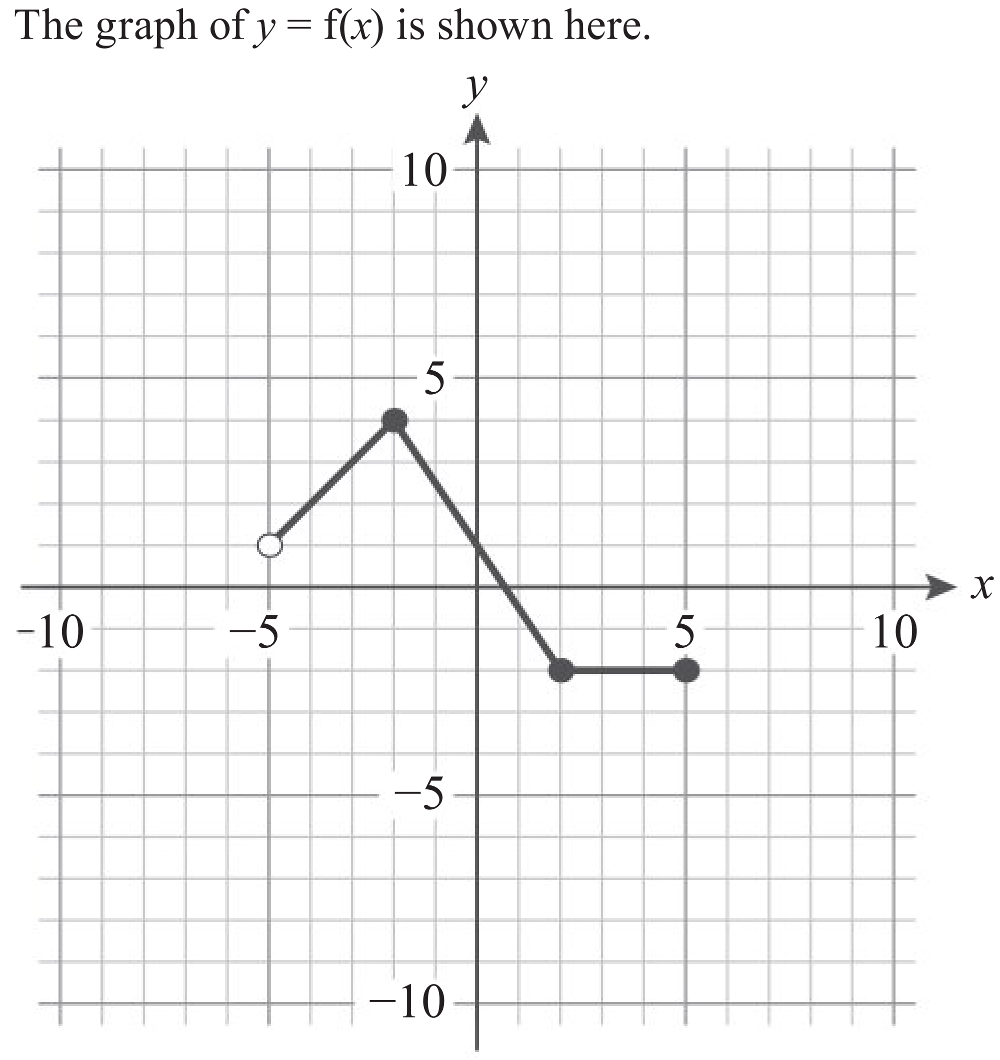
43 The graph of $y = 3x^2 + x - 2$ is stretched vertically with scale factor 2. Find the equation of the resulting graph in the form $y = ax^2 + bx + c$.

44 The graph of $y = f(x)$ is shown here.



Sketch the graph of $y = f(2x)$.

45 The graph of $y = f(x)$ is shown here.

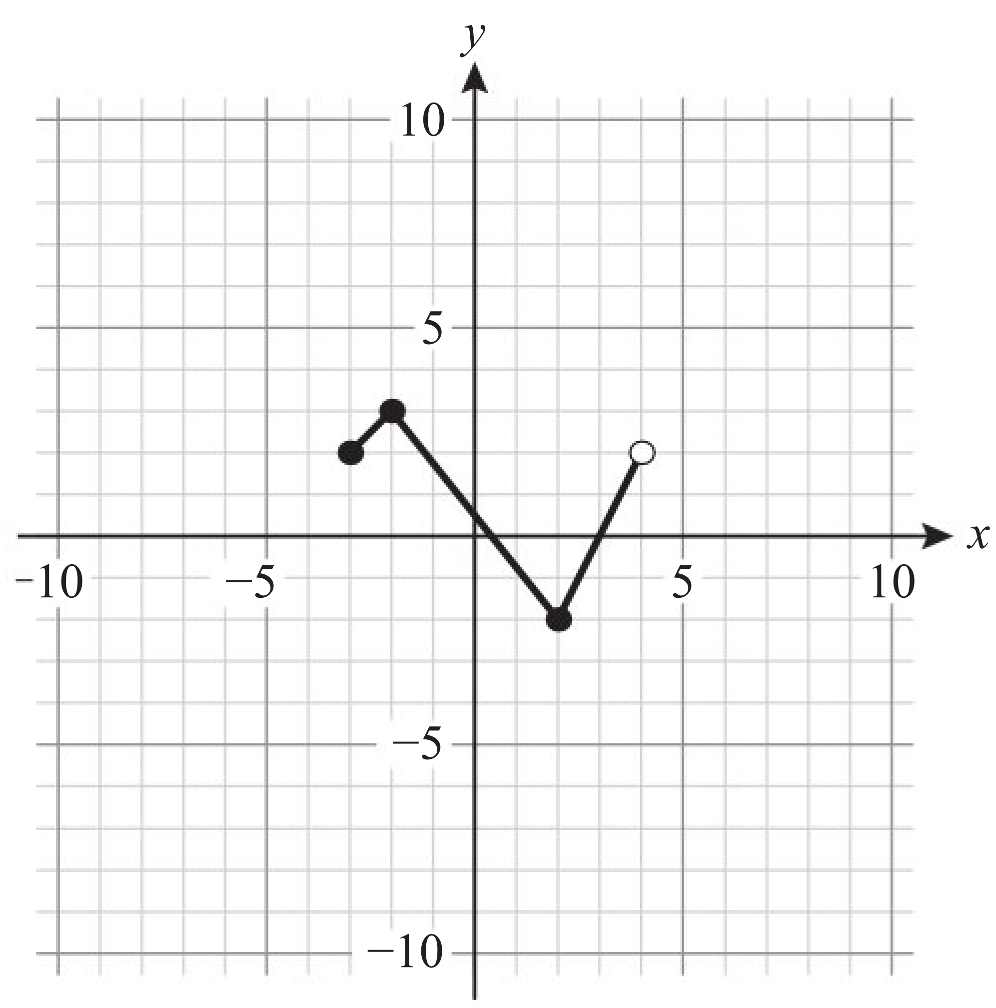


Sketch the graph of $y = -f(x)$.

46 The graph of $y = x^3 + 3x^2 - 4x + 1$ is reflected in the y-axis.
Find the equation of the resulting graph in the form $y = ax^3 + bx^2 + cx + d$.

47 The graph of $y = f(x)$ has a single vertex at $(3, -2)$.
Find the coordinates of the vertex on the graph $y = 4f(x) + 1$.

48 The graph of $y = f(x)$ is shown here.



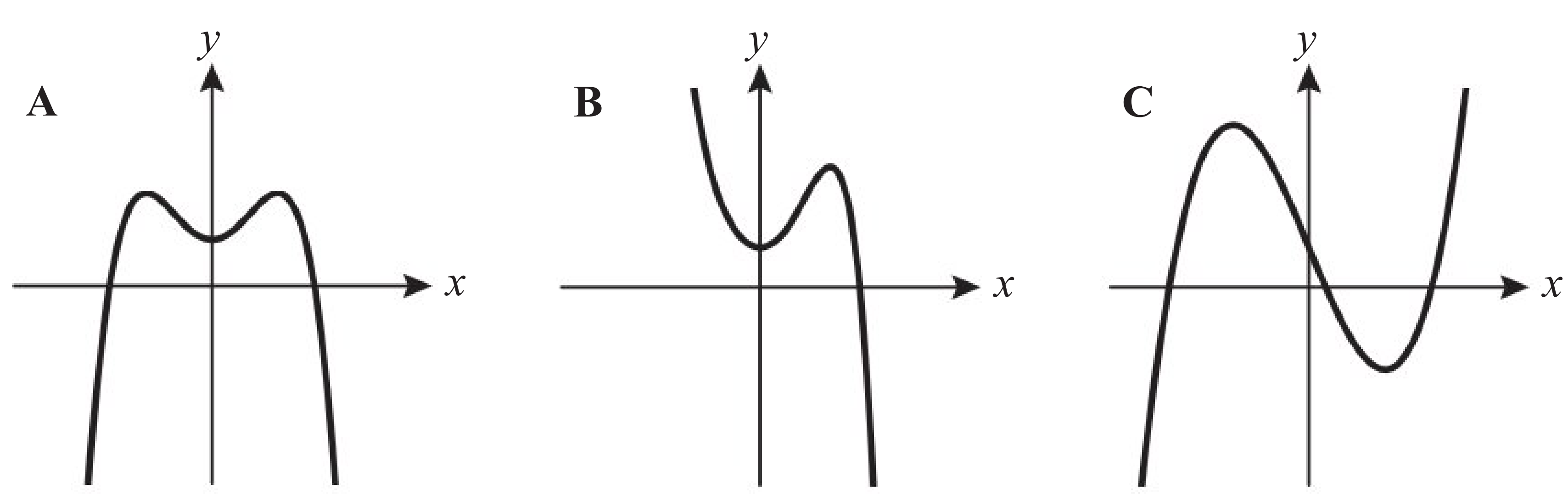
Sketch the graph of $y = 3f\left(\frac{1}{2}x\right)$.

49 Match each equation with the corresponding graph.

a $y = x^3 - 4x + 1$

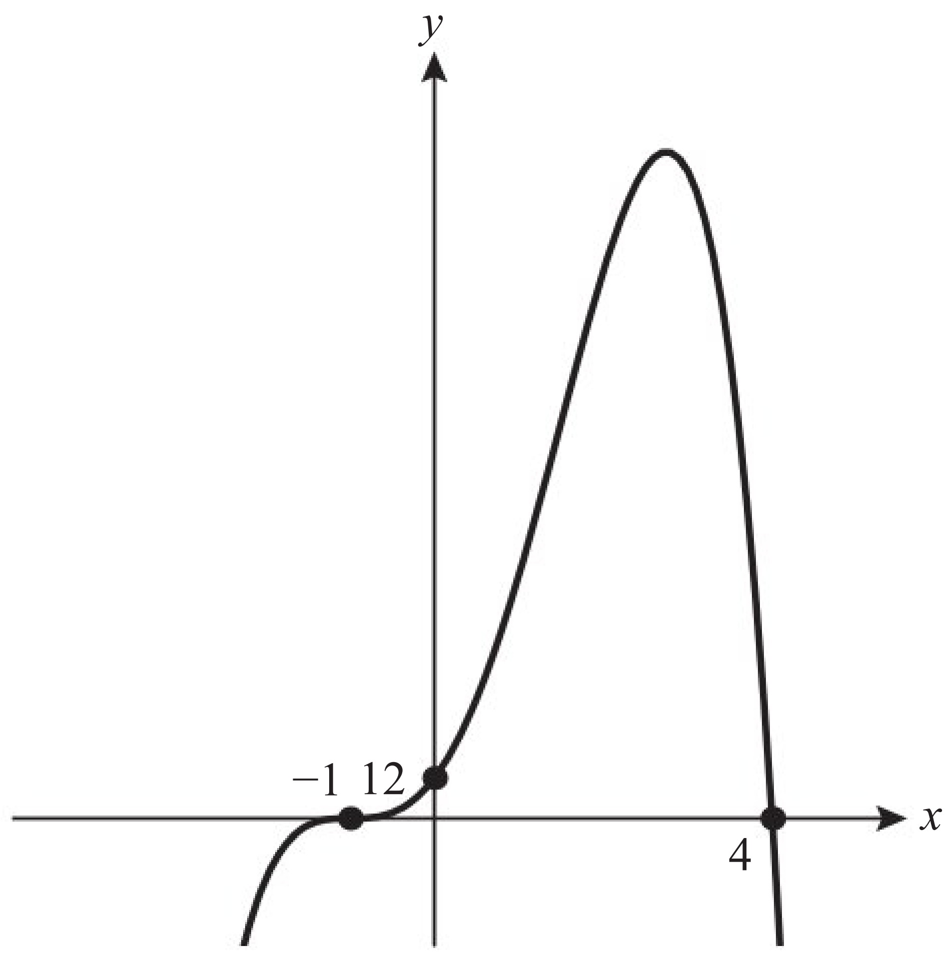
b $y = -x^4 + 2x^2 + 1$

c $y = -x^5 + 3x^2 + 1$



50 Sketch the graph of $y = -2(x - 1)^2(x + 3)$, labelling all axis intercepts.

51 Find a possible equation for the following graph:



52 Find the remainder when $f(x) = 2x^3 - x^2 + 4x + 3$ is divided by $(2x + 3)$.

53 Find the value of a such that $(3x - 4)$ is a factor of $f(x) = 3x^3 + 5x^2 - 42x + a$.

54 **a** Show that $(x + 2)$ is a factor of $f(x) = 3x^3 + 22x^2 + 20x - 24$.

b Hence solve $f(x) = 0$.

55 Find the sum and product of the roots of $f(x) = 6x^4 + 55x^3 + 147x^2 + 72x - 80$.

56 The equation $4x^2 - 2x + 3 = 0$ has roots α and β.
Find a quadratic equation with integer coefficients and roots $\frac{2}{\alpha}$ and $\frac{2}{\beta}$.

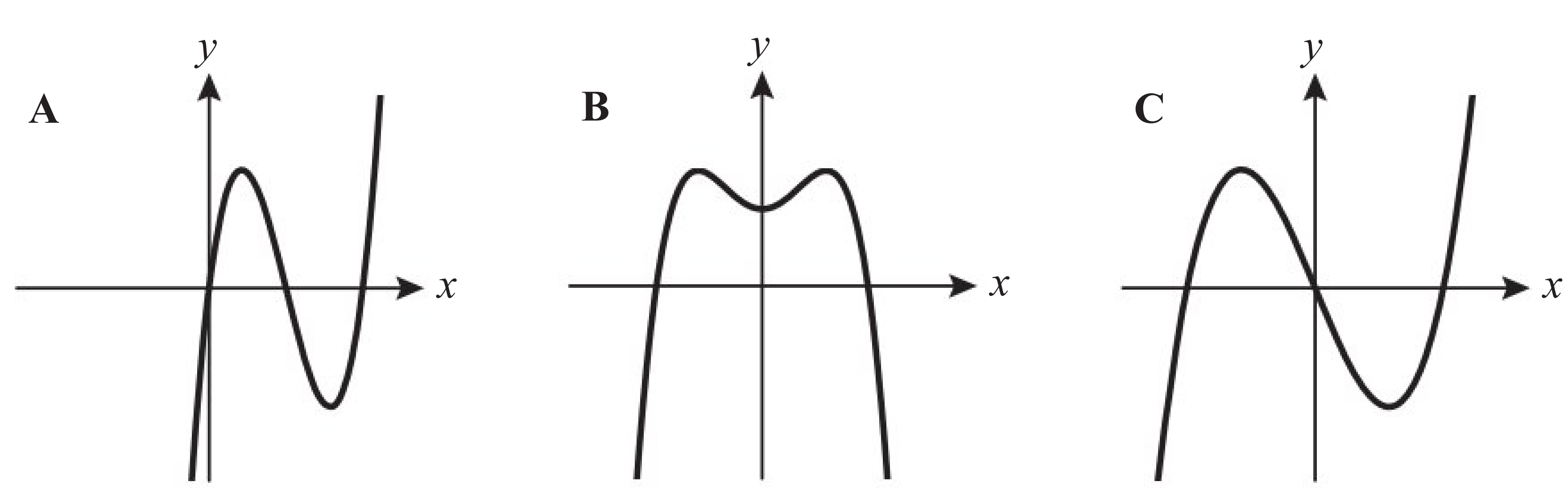
57 The equation $ax^4 - 3x^3 - 13x^2 + 37x + c = 0$ has roots $\frac{1}{2}$, -3, $2 + \mathrm{i}$ and $2 - \mathrm{i}$.
Find the values of a and c.

58 Sketch the graph of $y = \frac{2x + 3}{x^2 + 3x - 4}$, labelling all axis intercepts and asymptotes.

59 Sketch the graph of $y = \frac{x^2 + x - 12}{x - 1}$, labelling all axis intercepts and asymptotes.

60 Determine algebraically whether $f(x) = \frac{\cos x}{x}$, $x \neq 0$ is an odd function, an even function or neither.

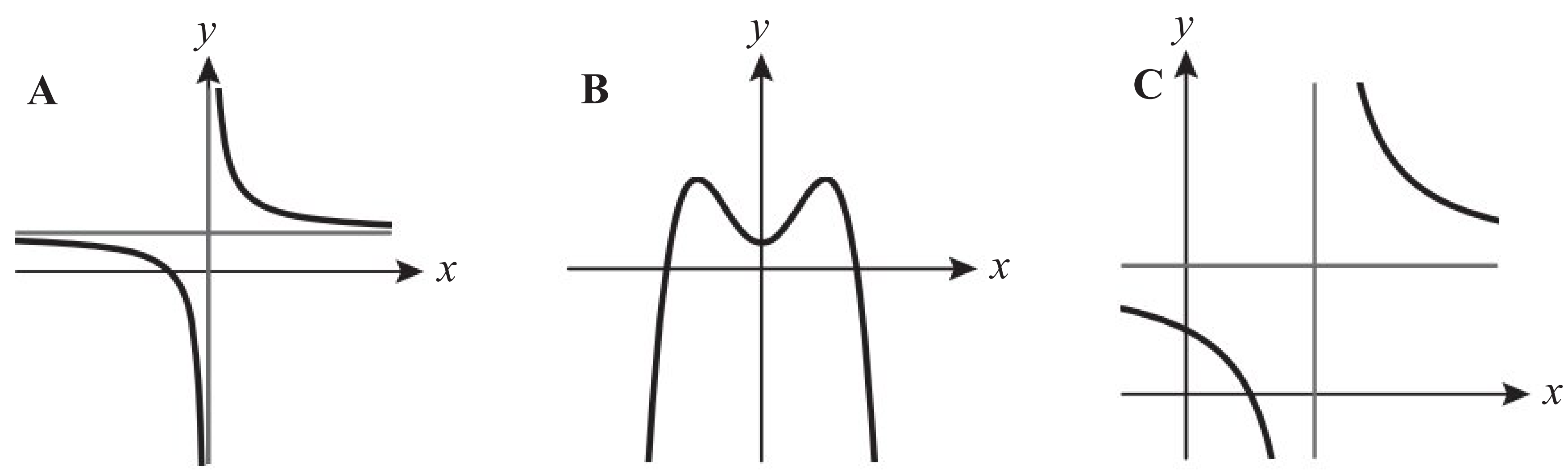
61 From their graphs, determine whether these functions are odd, even or neither.



62 The function f is defined by $f(x) = -x^3 - 3x^2 + 9x + 8$.
Find the largest interval of the form $a \leqslant x \leqslant b$ for which the inverse function f^{-1} exists.

63 Determine algebraically whether the function $f(x) = \dfrac{3x - 1}{x - 3}$ is self-inverse or not.

64 From their graphs, determine whether these functions are self-inverse or not.



65 Solve the inequality $x^3 \leqslant 10x - 3x^2$.

66 Solve the inequality $\dfrac{2x - 3}{x - 1} > e^{-x}$.

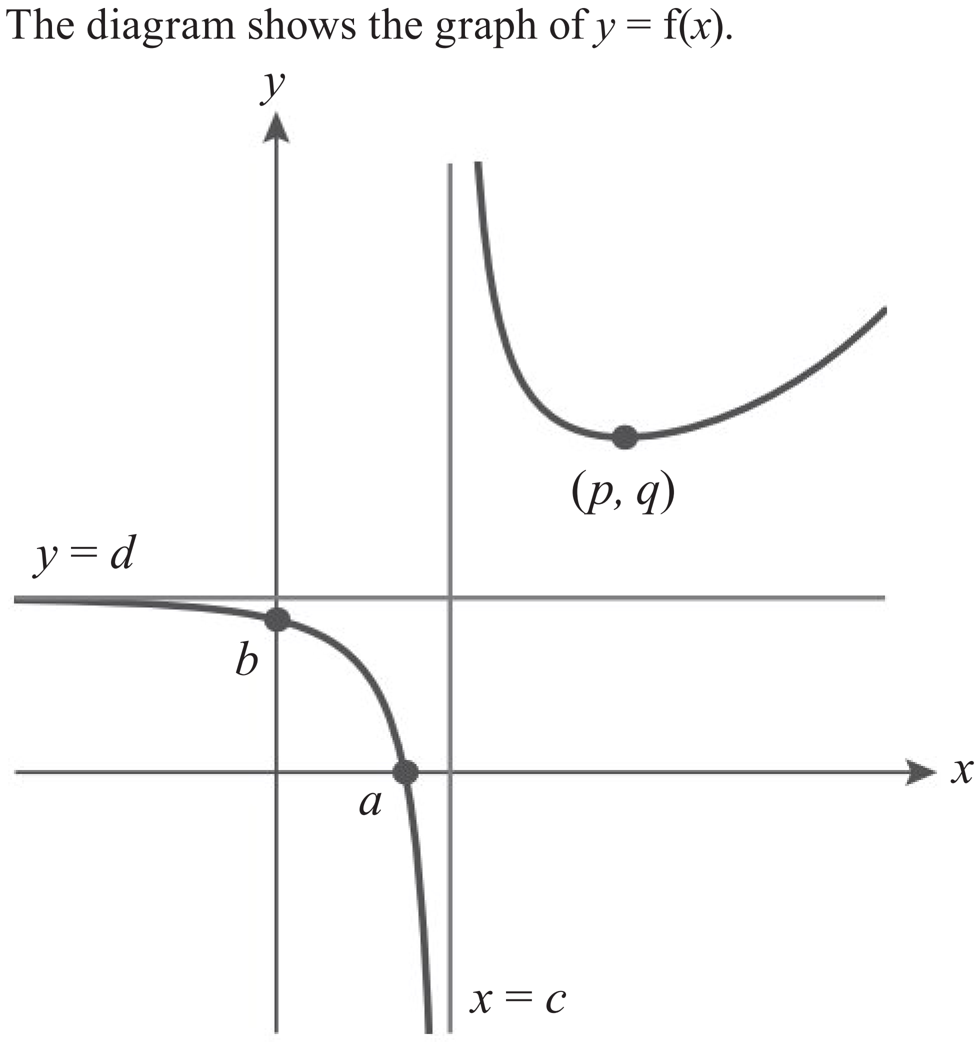
67 Sketch the graph of $y = |x^2 + x - 6|$.

68 Sketch the graph of $y = |x|^2 + |x| - 6$.

69 Solve the equation $|x + 1| = 4 - 3x$.

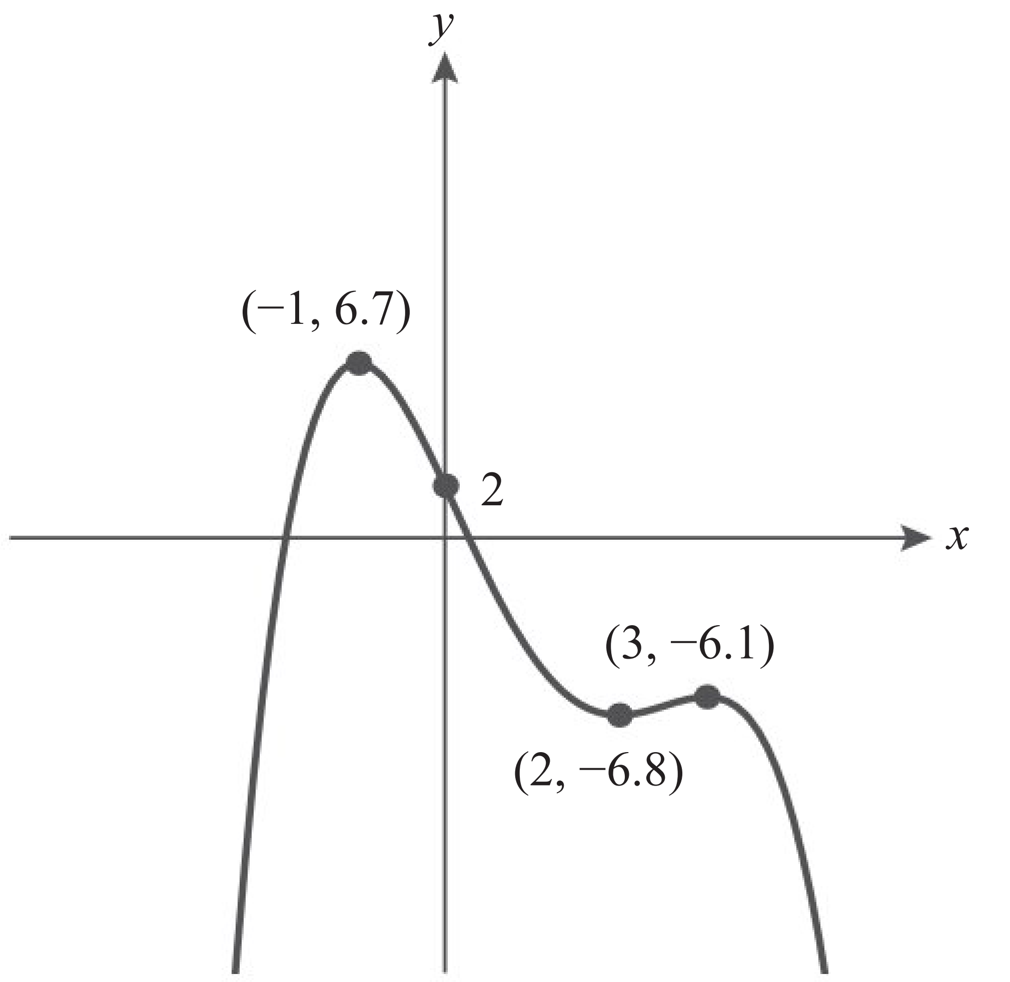
70 Solve the inequality $|x - 2| > |2x + 3|$.

71 The diagram shows the graph of $y = f(x)$.



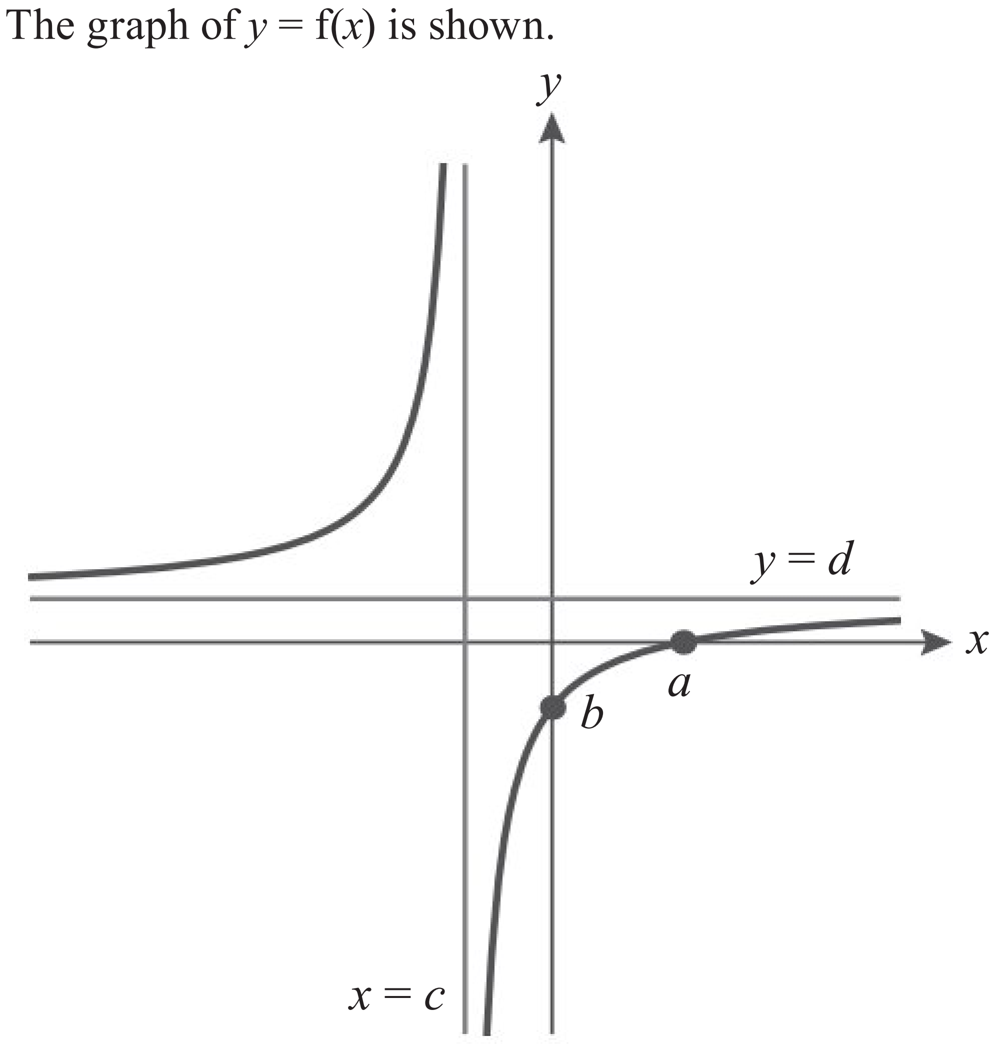
Sketch the graph of $y = \dfrac{1}{f(x)}$.

72 The graph of $y = \mathrm{f}(x)$ is shown.



Sketch the graph of $y = \mathrm{f}\left(\frac{1}{2}x - 3\right)$.

73 The graph of $y = \mathrm{f}(x)$ is shown.



Sketch the graph of $y = [\mathrm{f}(x)]^2$.

3 Geometry and trigonometry

Syllabus content

S3.1a	**Distance and midpoints**		
Book Section 4B	Revised ☐		
Syllabus wording	*You need to be able to:*	*Question*	
The distance between two points in three-dimensional space, and their midpoint.	Find the distance between points (x_1, y_1, z_1) and (x_2, y_2, z_2) using: $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$	1	☐
	Find the midpoint using $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$	2	☐

S3.1b	**Volume and surface area of 3D solids**		
Book Section 5A	Revised ☐		
Syllabus wording	*You need to be able to:*	*Question*	
Volume and area of three-dimensional solids.	Find the volume and surface area of a sphere using $V = \frac{4}{3}\pi r^3$ $A = 4\pi r^2$ where r is the radius.	3	☐
	Find the volume and curved surface area of a right cone using $V = \frac{1}{3}\pi r^2 h$ $A = \pi rl$ where r is the radius, h is the height and l is the slant height.	4	☐
	Find the volume and surface area of a right pyramid using $V = \frac{1}{3}Ah$ where A is the area of the base and h is the height.	5	☐
	Find the volume and surface area of combinations of solids.	6	☐

S3.1c	**Angle between intersecting lines and planes**		
Book Section 5B, 5C	Revised ☐		
Syllabus wording	*You need to be able to:*	*Question*	
The size of an angle between two intersecting lines or between a line and a plane.	Find the angle between two lines in two dimensions.	7	☐
	Find the angle between a line and a plane.	8	☐
	Find the angle between two intersecting lines in three dimensions.	9	☐

S3.2a	**Trigonometry in right-angled triangles**		
Book Section 5B	Revised ☐		
Syllabus wording	*You need to be able to:*	*Question*	
Use of sine, cosine and tangent ratios to find the sides and angles of right-angled triangles.	Find lengths and angles in right-angled triangles using the sine, cosine and tangent ratios.	10	☐

S3.2b	**Trigonometry in non-right-angled triangles**		
Book Section 5B	Revised ☐		
Syllabus wording	*You need to be able to:*	*Question*	
The sine rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.	Find lengths and angles in non-right-angled triangles using the sine rule.	11	☐
The cosine rule: $c^2 = a^2 + b^2 - 2ab\cos C$, $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$.	Find lengths and angles in non-right-angled triangles using the cosine rule.	12	☐
Area of a triangle as $\frac{1}{2}ab\sin C$.	Find the area of a triangle when you do not know the perpendicular height.	13	☐

S3.3	Applications of trigonometry		
	Book Section 5C	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Angles of elevation and depression.	Use trigonometry in questions involving angles of elevation and depression.	14	☐
Construction of labelled diagrams from written statements.	Construct diagrams from given information (often involving bearings) and solve using trigonometry.	15	☐

S3.4	Radian measure and applications to circles		
	Book Section 18A	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
The circle: radian measure of angles; length of an arc; area of a sector.	Convert between degrees and radians.	16	☐
	Find the length of an arc, $l = r\theta$, where r is the radius, θ is the angle measured in radians.	17	☐
	Find the area of a sector, $A = \frac{1}{2} r^2 \theta$ where r is the radius, θ is the angle measured in radians.	18	☐

S3.5	Extending definitions of trigonometric functions		
	Book Section 18B	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Definition of $\cos\theta$, $\sin\theta$ in terms of the unit circle.	Use the definition of $\cos\theta$ and $\sin\theta$ in terms of the unit circle.	19	☐
Definition of $\tan\theta$ as $\frac{\sin\theta}{\cos\theta}$.	Use the definition $\tan\theta = \frac{\sin\theta}{\cos\theta}$.	20	☐
Exact values of trigonometric ratios of $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ and their multiples.	Recall the exact values of sine, cosine and tangent of $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ and their multiples.	21	☐
Extension of the sine rule to the ambiguous case.	Use the sine rule to find two possible solutions for an angle in a triangle.	22	☐

S3.6	Trigonometric identities		
	Book Section 18C	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
The Pythagorean identity $\cos^2\theta + \sin^2\theta \equiv 1$.	Use the identity $\cos^2\theta + \sin^2\theta \equiv 1$.	23	☐
Double angle identities for sine and cosine.	Use the identity $\sin 2\theta \equiv 2\sin\theta\cos\theta$.	24	☐
	Use the identities $\cos 2\theta \equiv \cos^2\theta - \sin^2\theta \equiv 2\cos^2\theta - 1 \equiv 1 - 2\sin^2\theta$.	25	☐

S3.7	Graphs of trigonometric functions		
	Book Section 18D	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
The circular functions $\sin x$, $\cos x$ and $\tan x$; amplitude, their periodic nature, and their graphs.	Sketch the graphs of $y = \sin x$, $y = \cos x$ and $y = \tan x$.	26	☐
Transformations.	Sketch the graphs of transformations of trigonometric functions.	27	☐
Composite functions of the form $f(x) = a\sin(b(x + c)) + d$.	Sketch graphs of the form $y = a\sin(b(x + c)) + d$ and $y = a\cos(b(x + c)) + d$.	28	☐
Real-life contexts	Form trigonometric models from given information.	29	☐

S3.8	Book Section 18E	**Solving trigonometric equations** Revised ☐	
Syllabus wording		*You need to be able to:*	*Question*
Solving trigonometric equations in a finite interval, both graphically and analytically.		Solve trigonometric equations graphically using your GDC.	30 ☐
		Solve analytically trigonometric equations of the form $\sin\theta = k$, $\cos\theta = k$ and $\tan\theta = k$.	31 ☐
		Solve analytically trigonometric equations of the form $\sin A = k$, $\cos A = k$ and $A = k$, where $A = \mathrm{f}(\theta)$.	32 ☐
		Use identities to solve trigonometric equations.	33 ☐
Equations leading to quadratic equations in $\sin x$, $\cos x$ or $\tan x$.		Solve trigonometric equations as quadratics in a trigonometric function.	34 ☐

H3.9	Book Section 3A	**Reciprocal and inverse trigonometric functions** Revised ☐	
Syllabus wording		*You need to be able to:*	*Question*
Definition of the reciprocal trigonometric ratios $\sec\theta$, $\operatorname{cosec}\theta$ and $\cot\theta$.		Use the definitions of $\sec\theta$, $\operatorname{cosec}\theta$ and $\cot\theta$.	35 ☐
		Sketch the graphs of $y = \sec x$, $y = \operatorname{cosec} x$ and $y = \cot x$.	36 ☐
Pythagorean identities: $1 + \tan^2\theta \equiv \sec^2\theta$; $1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$.		Use the identities $1 + \tan^2\theta \equiv \sec^2\theta$ and $1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$.	37 ☐
The inverse functions $\mathrm{f}(x) = \arcsin x$, $\mathrm{f}(x) = \arccos x$, $\mathrm{f}(x) = \arctan x$; their domains and ranges; their graphs.		Use the definitions of $\arcsin x$, $\arccos x$ and $\arctan x$.	38 ☐
		Sketch the graphs of $y = \arcsin x$, $y = \arccos x$ and $y = \arctan x$.	39 ☐

H3.10	Book Section 3B	**Compound angle identities** Revised ☐	
Syllabus wording		*You need to be able to:*	*Question*
Compound angle identities.		Use the identities $\sin(A \pm B) \equiv \sin A \cos B \pm \cos A \sin B$, $\cos(A \pm B) \equiv \cos A \cos B \mp \sin A \sin B$ and $\tan(A \pm B) \equiv \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$.	40 ☐
Double angle identity for tan.		Use the identity $\tan 2\theta \equiv \dfrac{2\tan\theta}{1 - \tan^2\theta}$.	41 ☐

H3.11	Book Section 3B	**Symmetries of trigonometric graphs** Revised ☐	
Syllabus wording		*You need to be able to:*	*Question*
Relationships between trigonometric functions and the symmetry properties of their graphs.		Use compound angle identities to establish symmetry properties of trigonometric graphs.	42 ☐

H3.12a	Book Section 8A	**Introduction to vectors** Revised ☐	
Syllabus wording		*You need to be able to:*	*Question*
Representation of vectors using directed line segments.		Express vectors given as directed line segments in 2D as column vectors.	43 ☐
Base vectors **i**, **j**, **k**.		Express column vectors in terms of the base vectors **i**, **j**, **k**.	44 ☐
Components of a vector: $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$.		Express vectors given in terms of base vectors as column vectors.	45 ☐
Algebraic and geometric approaches to the following: • the sum and difference of two vectors • the zero vector **0**, the vector $-\mathbf{v}$ • multiplication by a scalar, $k\mathbf{v}$, parallel vectors • magnitude of a vector, $\lvert\mathbf{v}\rvert$; unit vectors $\dfrac{\mathbf{v}}{\lvert\mathbf{v}\rvert}$.		Add and subtract vectors algebraically and geometrically.	46 ☐
		Multiply vectors by scalars.	47 ☐
		Determine whether vectors are parallel.	48 ☐
		Calculate the magnitude of a vector using $\lvert\mathbf{v}\rvert = \sqrt{v_1^2 + v_2^2 + v_3^2}$	49 ☐
		Find a unit vector in a given direction.	50 ☐

H3.12b Geometry and vectors

Book Section 8B	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*
Position vectors $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$. Displacement vector $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.	Find the displacement vector between two points.	51 ☐
	Find the distance between two points.	52 ☐
Proofs of geometrical properties using vectors.	Prove geometrical properties using vectors.	53 ☐

H3.13 The scalar product

Book Section 8C	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*
The definition of the scalar product of two vectors.	Calculate the scalar product of two vectors using the definition $\mathbf{v} \cdot \mathbf{w} = v_1w_1 + v_2w_2 + v_3w_3$.	54 ☐
	Calculate the scalar product of two vectors using the definition $\mathbf{v} \cdot \mathbf{w} = \lvert\mathbf{v}\rvert\lvert\mathbf{w}\rvert \cos\theta$ where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$.	55 ☐
	Use the algebraic properties of the scalar product.	56 ☐
The angle between two vectors.	Find the angle between two vectors using the result $\cos\theta = \frac{v_1w_1 + v_2w_2 + v_3w_3}{\lvert\mathbf{v}\rvert\lvert\mathbf{w}\rvert}$.	57 ☐
Perpendicular vectors; parallel vectors.	Use that fact that if $\mathbf{v}$ and $\mathbf{w}$ are perpendicular then $\mathbf{v} \cdot \mathbf{w} = 0$ and if $\mathbf{v}$ and $\mathbf{w}$ are parallel then $\lvert\mathbf{v} \cdot \mathbf{w}\rvert = \lvert\mathbf{v}\rvert\lvert\mathbf{w}\rvert$.	58 ☐

H3.14 Equation of a line in 3D

Book Section 8D	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*
Vector equation of a line in two and three dimensions: $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$.	Find the vector equation of a line given a point on the line and a vector parallel to the line: $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$.	59 ☐
	Find the vector equation of a line given two points on the line.	60 ☐
	Convert between the vector form and the parametric form of the equation of a line: $x = x_0 + \lambda l, y = y_0 + \lambda m, z = z_0 + \lambda n$.	61 ☐
	Convert between the vector form and the Cartesian form of the equation of a line: $\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$	62 ☐
The angle between two lines.	Find the angle between two lines.	63 ☐
Simple applications to kinematics.	Model the motion of a particle moving with a constant velocity vector.	64 ☐

H3.15 Intersection of lines

Book Section 8E	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*
Coincident, parallel, intersecting and skew lines, distinguishing between these cases.	Determine whether two lines are parallel or coincident.	65 ☐
	Determine whether two non-parallel lines intersect or are skew.	66 ☐
Points of intersection.	Find the point of intersection of two intersecting lines.	67 ☐

H3.16	Book Section 8F	**The vector product** Revised ☐	
Syllabus wording		*You need to be able to:*	*Question*
The definition of the vector product of two vectors.		Calculate the vector product of two vectors using the definition $\mathbf{v} \times \mathbf{w} = \begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix}$.	68 ☐
		Calculate the magnitude of the vector product of two vectors using the definition $\lvert\mathbf{v} \times \mathbf{w}\rvert = \lvert\mathbf{v}\rvert\lvert\mathbf{w}\rvert \sin\theta$ where θ is the angle between **v** and **w**.	69 ☐
Properties of the vector product.		Use the algebraic properties of the vector product.	70 ☐
Geometric interpretation of $\lvert\mathbf{v} \times \mathbf{w}\rvert$.		Calculate the area of a parallelogram with adjacent sides **v** and **w** using the formula $A = \lvert\mathbf{v} \times \mathbf{w}\rvert$.	71 ☐

H3.17	Book Section 8G	**Equation of a plane** Revised ☐	*Question*
Vector equation of a plane: $\mathrm{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, where **b** and **c** are non-parallel vectors within the plane.		Find the vector equation of a plane given a point on the plane and two vectors parallel to the plane: $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$.	72 ☐
		Find the vector equation of a plane given three points on the plane.	73 ☐
$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$, where **n** is a normal to the plane and **a** is the position vector of a point on the plane.		Find the equation of a plane in scalar product form given a point on the plane and a normal vector to the plane: $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$.	74 ☐
		Convert between vector form and scalar product form of the equation of a plane.	75 ☐
Cartesian equation of a plane $ax + by + cz = d$.		Find the Cartesian equation of a plane: $ax + by + cz = d$.	76 ☐

H3.18	Book Section 8H	**Angles and intersections between lines and planes** Revised ☐	
Syllabus wording		*You need to be able to:*	*Question*
Intersection of: • a line with a plane; • two planes; • three planes.		Find the point of intersection of a line and a plane.	77 ☐
		Find the line of intersection of two planes.	78 ☐
		Find the point or line of intersection of three intersecting planes.	79 ☐
		Determine the configuration of three non-intersecting planes.	80 ☐
Angle between: a line and a plane; two planes.		Find the angle between a line and a plane.	81 ☐
		Find the angle between two planes.	82 ☐

Practice questions

1 Find the distance between (2, −4, 5) and (7, 3, −1).

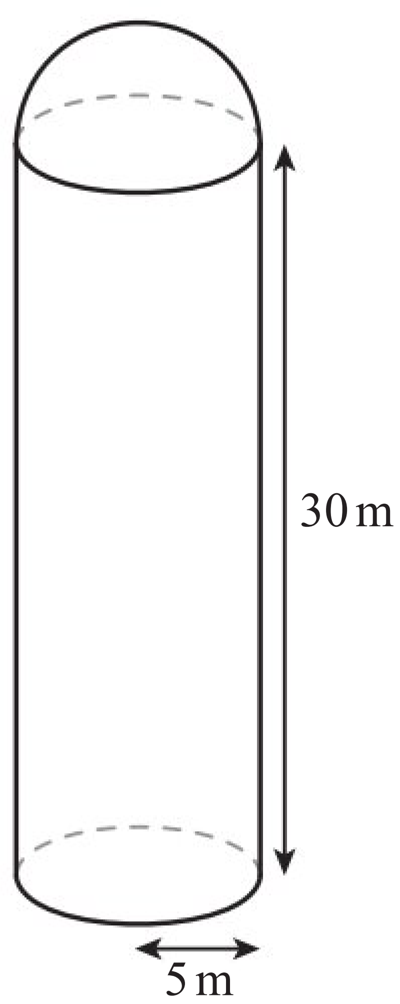
2 Find the midpoint of (1, 8, −3) and (−5, 2, 4).

3 Find, to three significant figures, the volume and surface area of a sphere of diameter 16 cm.

4 Find, to three significant figures, the volume and surface area of a cone with base radius 6 cm and height 15 cm.

5 Find, to three significant figures, the volume and surface area of a square-based pyramid with base side 5 cm and height 9 cm.

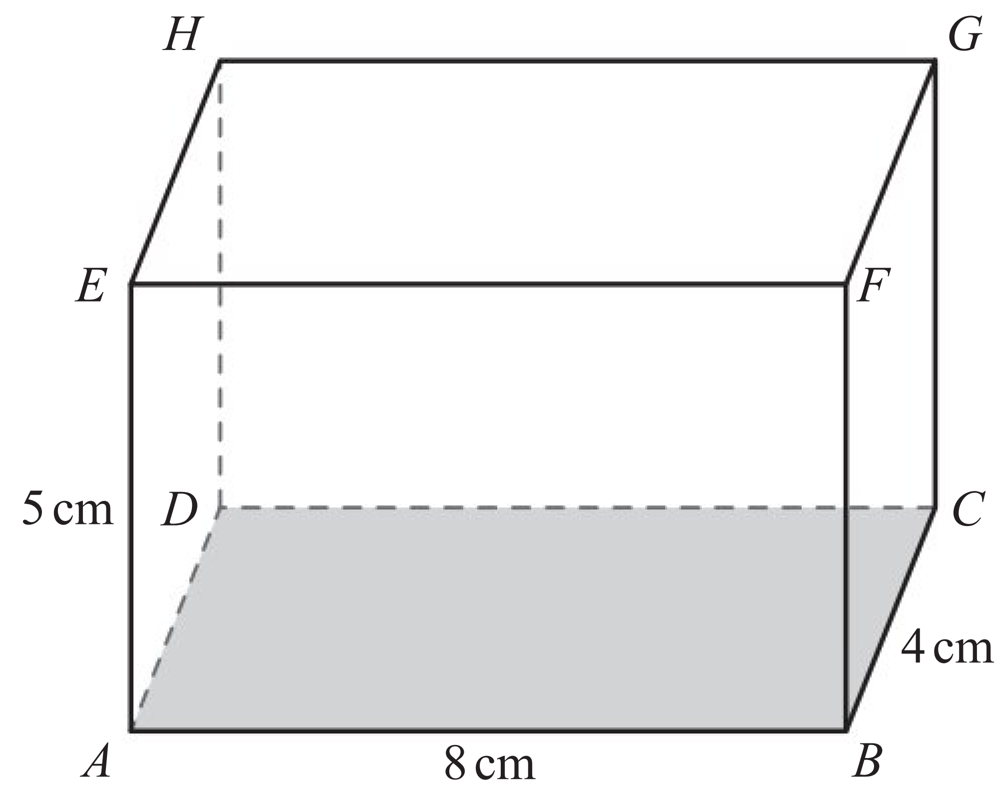
6 A grain silo is formed of a hemisphere on top of a cylinder of radius 5 m and height 30 m as shown:



Find the silo's volume.

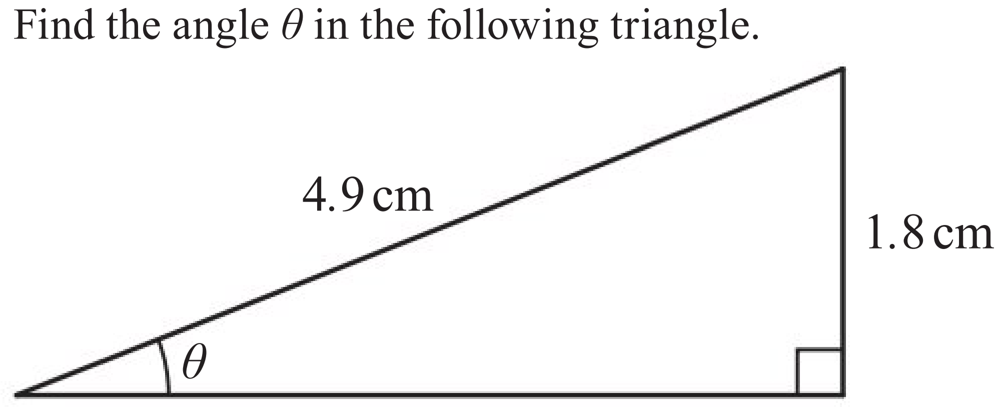
7 Find the acute angle between the lines $y = 4x - 3$ and $y = 5 - 3x$.

8 Find the angle between the line AG and the base plane $ABCD$.

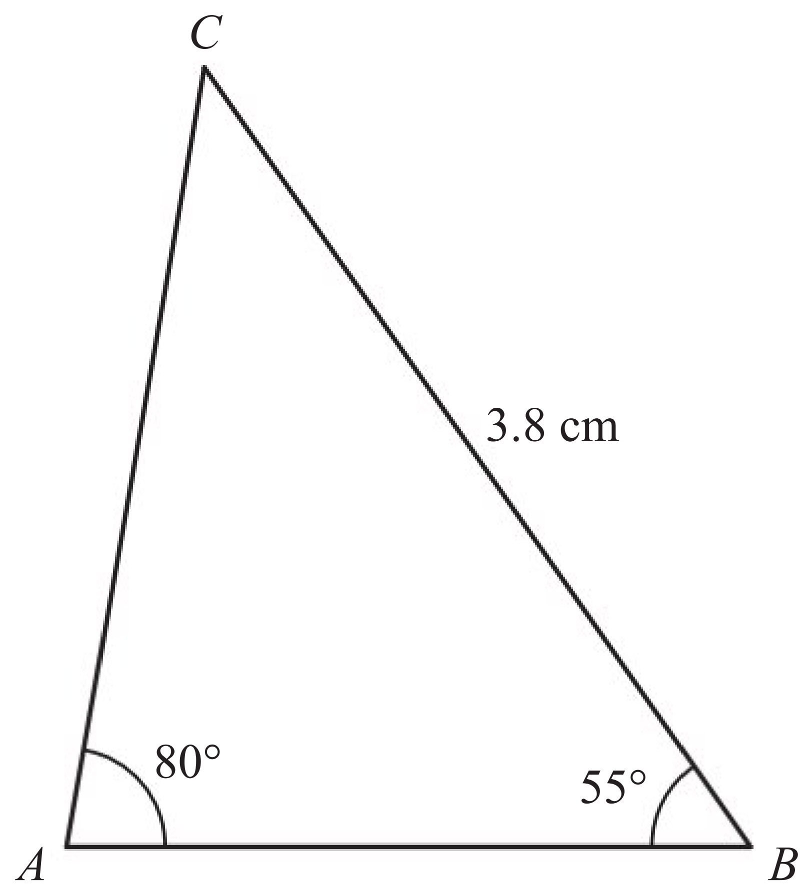


9 Find the acute angle between the diagonals AG and EC in the cuboid from question **8**.

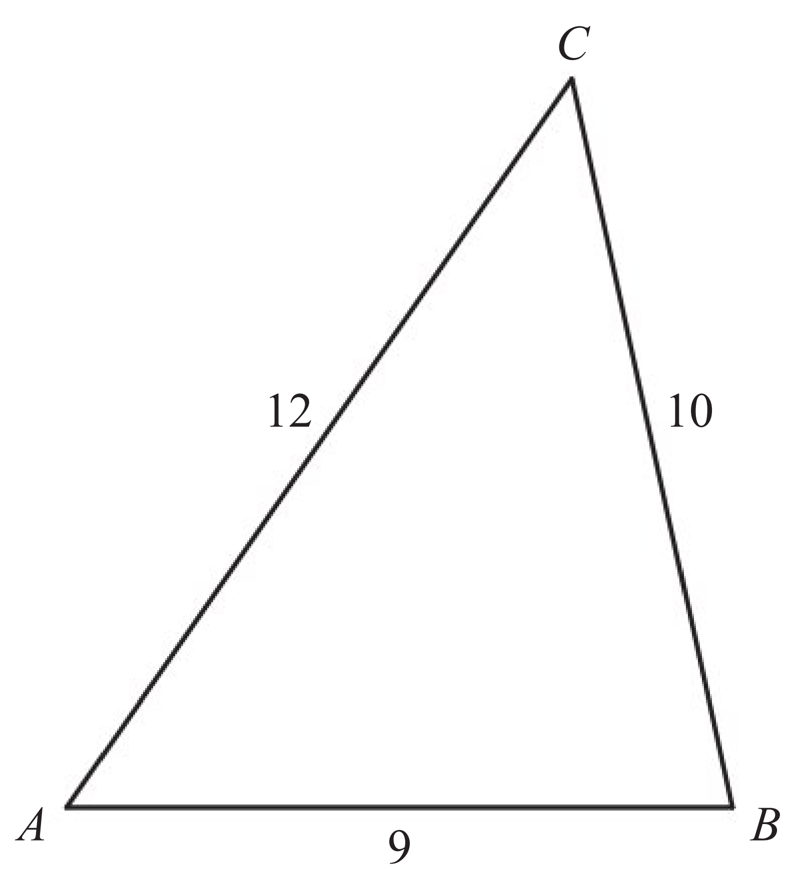
10 Find the angle θ in the following triangle.



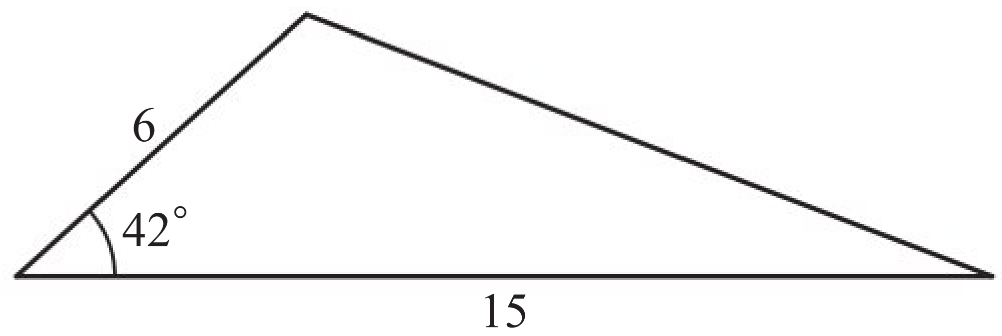
11 Find the length AC in the following triangle.



12 Find the angle C in the following triangle.



13 Find the area of the following triangle.



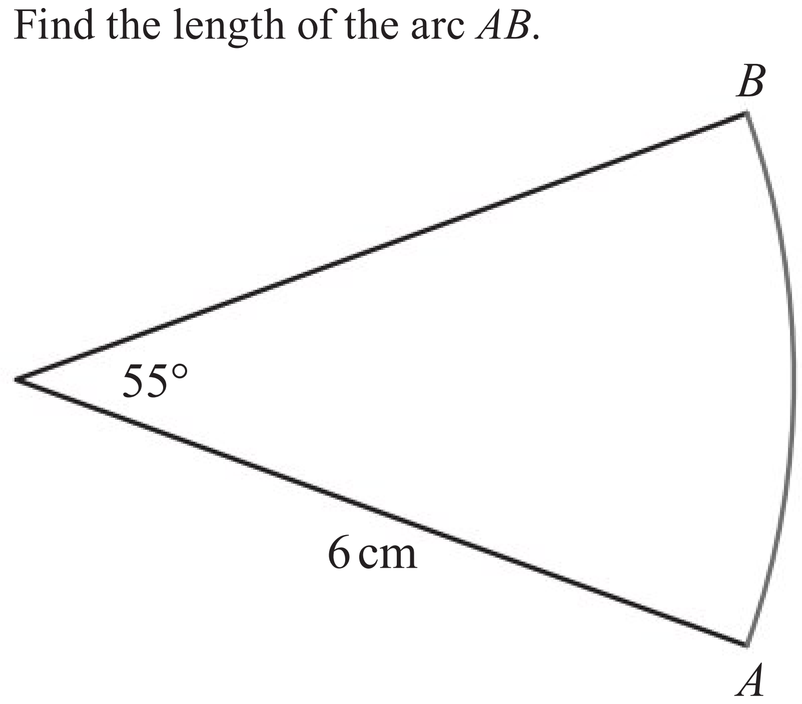
14 The angle of elevation of the top of a tree at a distance of 6.5 m is 68°.
Find the height of the tree.

15 A ship leaves port on a bearing of 030° and travels 150 km before docking. It then travels on a bearing of 110° for 80 km before docking again.
Find the distance it must now travel to return to where it started.

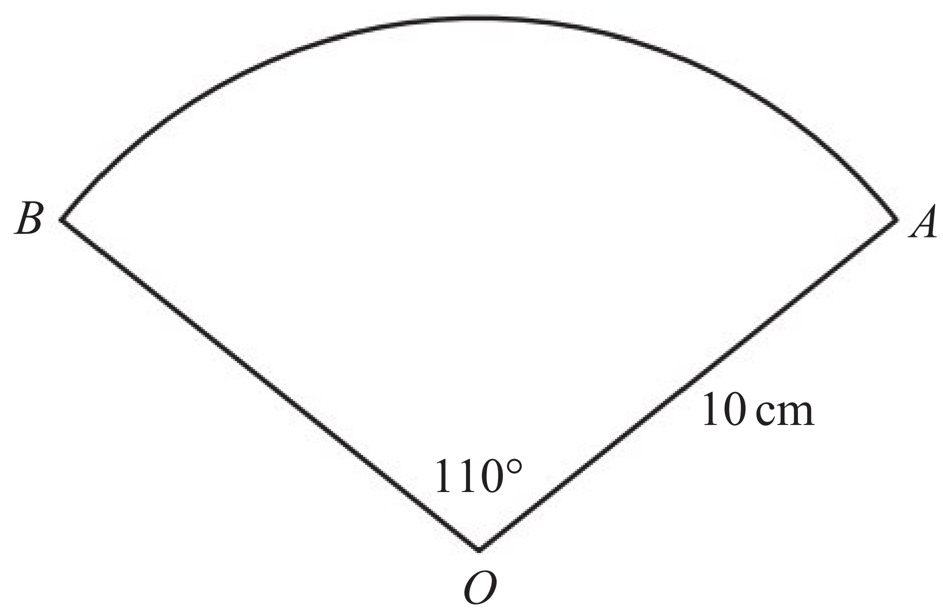
16 a Convert 55° to radians.

b Convert 1.2 radians to degrees.

17 Find the length of the arc *AB*.



18 Find the area of the sector *AOB*.



19 Given that $\sin\theta = 0.4$, find the value of

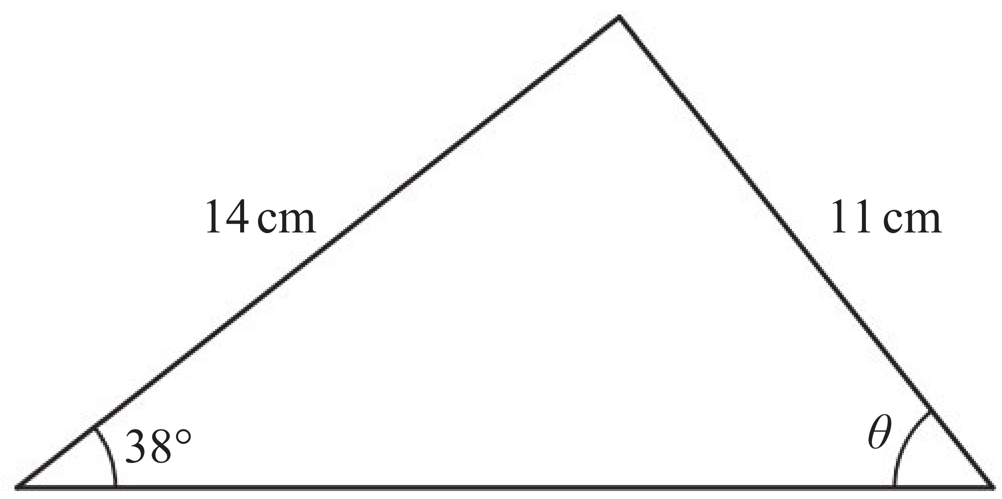
a $\sin(\theta + \pi)$

b $\cos\left(\theta - \frac{\pi}{2}\right)$.

20 Using the definition of the tangent function, show that $\tan(2\pi - \theta) = -\tan\theta$.

21 Find the exact value of $\cos\frac{4\pi}{3}$.

22 Find the size of the angle θ in the following triangle.



23 Given that $\sin\theta = \frac{3}{4}$, where $\frac{\pi}{2} < \theta < \pi$, find the exact value of $\cos\theta$.

24 Prove that $(\cos\theta + \sin\theta)^2 \equiv \sin 2\theta + 1$.

25 Find the exact value of $\cos 15°$.

26 Sketch the graph of $y = \cos x$ for $-2\pi \leqslant x \leqslant 2\pi$, stating the period and amplitude.

27 Sketch the graph of $y = \tan\frac{x}{2}$ for $-360° \leqslant x \leqslant 360°$, labelling all axis intercepts and asymptotes.

28 Sketch the graph of $y = 3\sin\left(2\left(x - \frac{\pi}{3}\right)\right) + 5$ for $0 \leqslant x \leqslant 2\pi$.

29 A particle P, on the top of a spring which is fixed to a table, is initially at its maximum height above the table of 0.5 m. Its minimum height above the table, 0.16 m, occurs 0.3 s later.
Find a model of the form $h = a\cos bt + c$ for the height, h m, of P above the table at time t seconds.

30 Solve the equation $7\cos\left(2x - \frac{\pi}{5}\right) = 4$ for $0 < x < \pi$.

31 Solve the equation $\tan\theta = -\frac{\sqrt{3}}{3}$ for $-\pi < \theta < \pi$.

32 Solve the equation $\sin(x + 75°) = \frac{\sqrt{3}}{2}$ for $0° < x < 360°$.

33 Solve the equation $\sin 2\theta = \cos\theta$ for $0 \leqslant \theta \leqslant 2\pi$.

34 Solve the equation $2\sin^2 x - 3\cos x - 3 = 0$ for $-180° \leqslant x \leqslant 180°$.

35 Evaluate $\sec \frac{\pi}{4}$.

36 Sketch the graph of $y = \operatorname{cosec} x$ for $-2\pi \leqslant x \leqslant 2\pi$.

37 Given that $\tan \theta = -\frac{2}{3}$, where $\frac{3\pi}{2} < \theta < 2\pi$, find the exact value of $\sin \theta$.

38 Evaluate $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$.

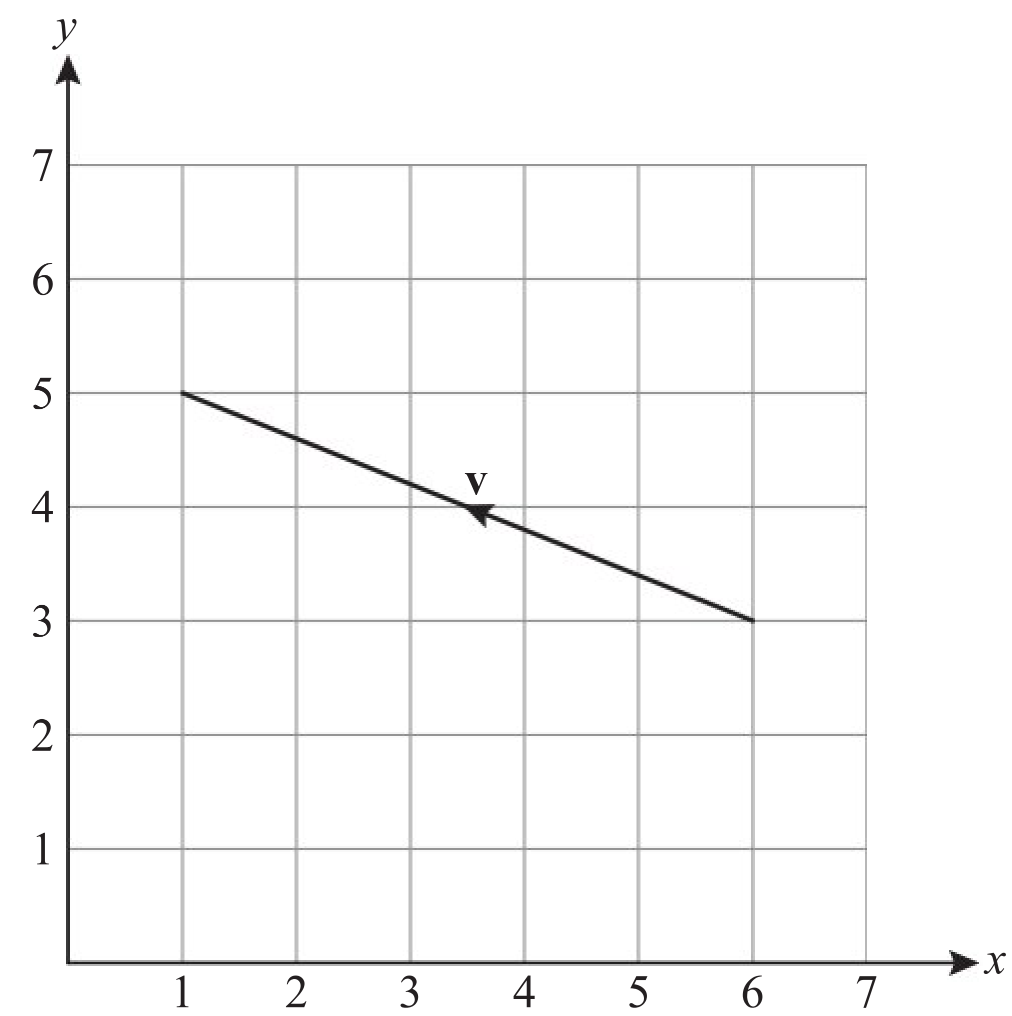
39 Sketch the graph of $y = \arctan x$, stating its domain and range.

40 Find the exact value of $\sin 105°$.

41 Given that $\tan 2\theta = -\frac{4}{3}$, find the possible values of $\tan \theta$.

42 Use a compound angle identity to simplify $\cos(\pi - x)$.

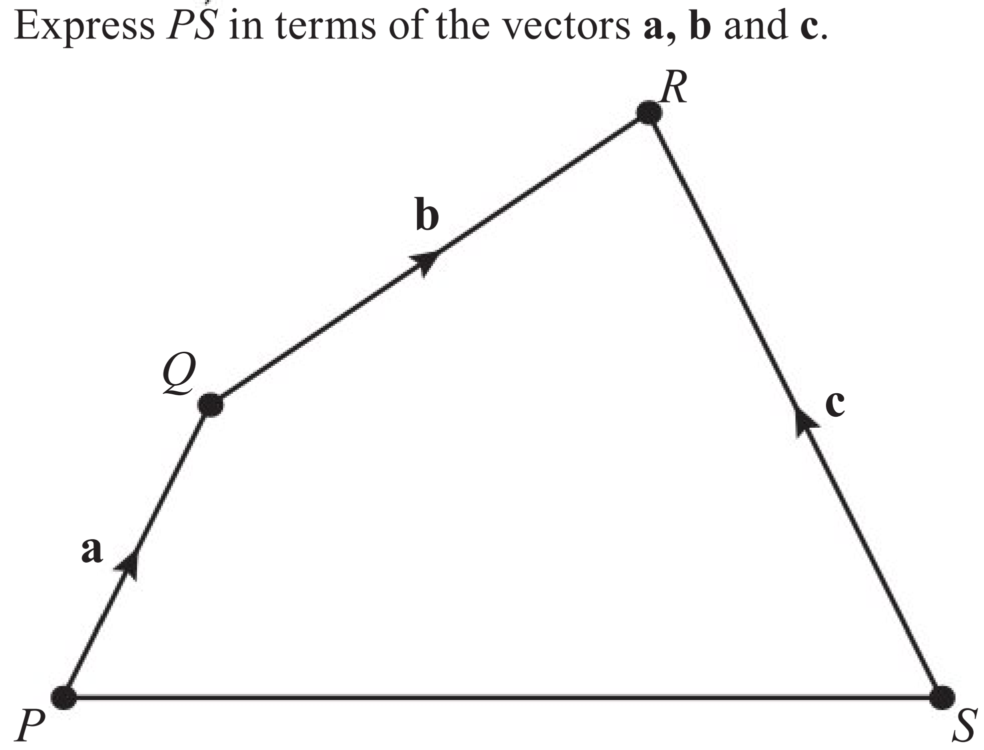
43 Express the following as a column vector:



44 Express the vector $\begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$ in terms of base vectors.

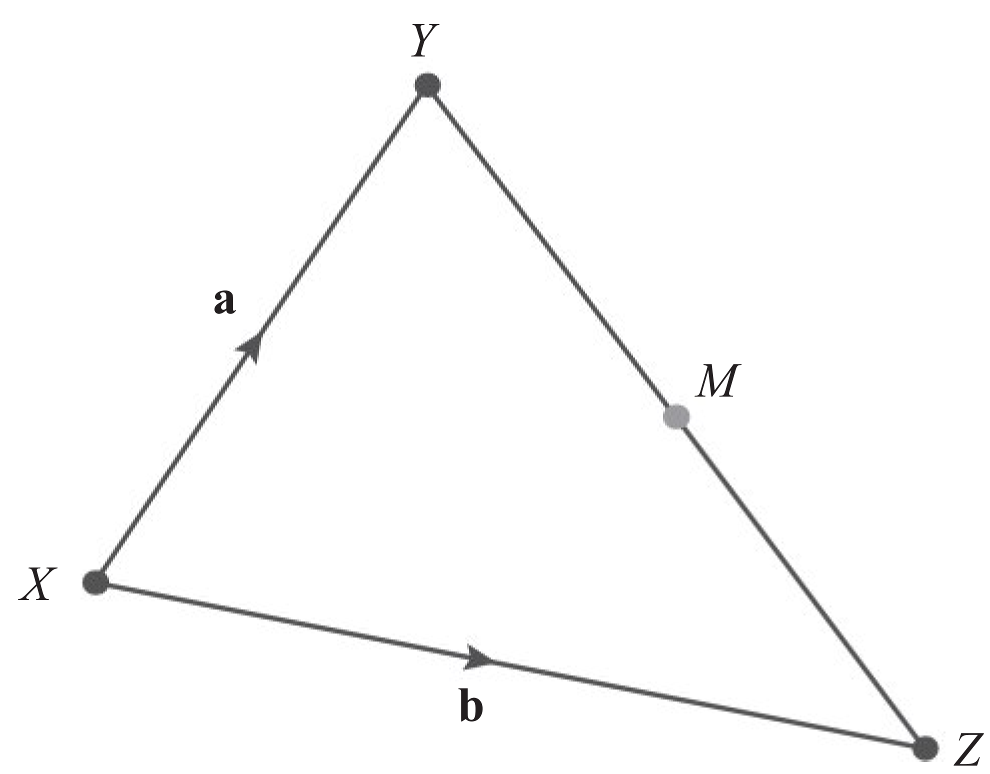
45 Express the vector $\mathbf{i} - 6\mathbf{k}$ as a column vector.

46 Express $\overrightarrow{PS}$ in terms of the vectors **a, b** and **c**.



47 M is the midpoint of YZ.

Find an expression for $\overrightarrow{XM}$ in terms of the vectors **a** and **b.**



48 Given that $\mathbf{a} = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} p \\ q \\ -12 \end{pmatrix}$ and that **a** and **b** are parallel, find the values of p and q.

49 Find the magnitude of the vector $\mathbf{v} = 3\mathbf{i} - 5\mathbf{j} - \mathbf{k}$.

50 Find a unit vector in the direction of $\begin{pmatrix} 2 \\ 6 \\ -3 \end{pmatrix}$.

51 Points A and B have coordinates (5, 1, −2) and (−4, 3, −1).
Find the displacement vector $\overrightarrow{AB}$.

52 The points A and B have position vectors $\mathbf{a} = \mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 5\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$.
Find the distance AB.

53 The points A, B, C and D have position vectors.
$\mathbf{a} = 3\mathbf{i} - 2\mathbf{k}$, $\mathbf{b} = 5\mathbf{i} - 4\mathbf{j} + \mathbf{k}$, $\mathbf{c} = 6\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$ and $\mathbf{d} = 4\mathbf{i} + \mathbf{j} + \mathbf{k}$.
Determine whether $ABCD$ is a parallelogram.

54 Given that $\mathbf{a} = \begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}$, calculate $\mathbf{a} \cdot \mathbf{b}$.

55 The vectors **a** and **b** are such that $|\mathbf{a}| = 3$, $|\mathbf{b}| = 8$ and the acute angle between **a** and **b** is 45°. Find the value of $\mathbf{a} \cdot \mathbf{b}$.

56 Find the acute angle between the vectors $\mathbf{a} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} - 2\mathbf{j} - 7\mathbf{k}$.

57 Given that $|\mathbf{a}| = 5$ and $|\mathbf{b}| = 2$, simplify $(2\mathbf{a} + 3\mathbf{b}) \cdot (2\mathbf{a} - 3\mathbf{b})$.

58 Find the value of t such that the vectors $\begin{pmatrix} 2+t \\ -3 \\ t \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 4t-1 \\ 5 \end{pmatrix}$ are perpendicular.

59 Find a vector equation of the line parallel to the vector $\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$ passing through the point (5, 4, −7).

60 Find the vector equation of the line through the points $A(-2, 6, 1)$ and $B(3, -5, 4)$.

61 Find the parametric equation of the line with vector equation $\mathbf{r} = \begin{pmatrix} -3 \\ 0 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$.

62 Find a vector equation of the line with Cartesian equation $\frac{x-4}{-3} = y + 2 = \frac{z-6}{5}$.

63 Find the acute angle between the lines with equations

$$\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \text{ and } \mathbf{r} = \begin{pmatrix} -4 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}.$$

64 An object moves with constant velocity $(3\mathbf{i} - \mathbf{j} + 2\mathbf{k})\ \mathrm{m\,s^{-1}}$. Initially it is at the point with position vector $(-5\mathbf{i} + \mathbf{k})$ m.

a Find the speed of the object.

b Find the position vector of the object after 10 seconds.

65 l_1: $\mathbf{r} = \begin{pmatrix} 0 \\ 8 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix}$

l_2: $\mathbf{r} = \begin{pmatrix} 6 \\ -7 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} -4 \\ 10 \\ -6 \end{pmatrix}$.

Show that the lines l_1 and l_2 have the same direction and determine whether they are coincident or parallel.

66 l_1: $\mathbf{r} = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

l_2: $\mathbf{r} = \begin{pmatrix} -2 \\ 13 \\ 11 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix}$.

Determine whether the non-parallel lines l_1 and l_2 intersect or are skew.

67 l_1: $\mathbf{r} = \begin{pmatrix} -1 \\ -7 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix}$

l_2: $\mathbf{r} = \begin{pmatrix} 2 \\ -3 \\ -6 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

Show that the lines l_1 and l_2 intersect and find the coordinates of their point of intersection.

68 Find $\begin{pmatrix}1\\4\\3\end{pmatrix} \times \begin{pmatrix}5\\2\\-2\end{pmatrix}$.

69 The vectors **a** and **b** are such that $|\mathbf{a}| = 4$, $|\mathbf{b}| = 5$ and the acute angle between **a** and **b** is 30°.
Find the magnitude of $\mathbf{a} \times \mathbf{b}$.

70 Given that $|\mathbf{a}| = 3$ and $|\mathbf{b}| = 4$, simplify $(2\mathbf{a} + \mathbf{b}) \times (5\mathbf{a} + 3\mathbf{b})$.

71 Find the area of the triangle with vertices $P(-1, 3, -2)$, $Q(4, -2, 5)$ and $R(1, 0, 3)$.

72 Find a vector equation of the plane parallel to the vectors $\begin{pmatrix}-3\\0\\2\end{pmatrix}$ and $\begin{pmatrix}4\\-5\\1\end{pmatrix}$ passing through the point $(1, -3, 6)$.

73 Find the vector equation of the plane through the points $A(4, -6, 0)$, $B(3, 1, 4)$ and $C(-2, 2, 1)$.

74 A plane has normal $\begin{pmatrix}-3\\1\\2\end{pmatrix}$ and contains the point $(5, -8, 4)$.

Find the equation of the plane in scalar product form.

75 A plane has vector equation $\mathbf{r} = \begin{pmatrix} 9 \\ 2 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix}$.

Find the equation of the plane in scalar product form.

76 A plane has normal $\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$ and contains the point (3, 0, 5).

Find the equation of the plane in Cartesian form.

77 Find the point of intersection of the line $\mathbf{r} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 1 \\ -6 \end{pmatrix}$ and the plane $2x - 4y + z = 16$.

78 Find the equation of the line of intersection of the planes $x + 2y - 3z = 5$ and $4x - y + 2z = 11$.

79 Find the coordinates of the point of intersection of the planes:
Π_1: $x + 3y - 2z = 11$
Π_2: $2x - 4y + z = -3$
Π_3: $4x + 2y - 5z = 21$.

80 Π_1: $x - 3y + 2z = -7$
Π_2 : $4x + y - z = -5$
Π_3: $6x - 5y + 3z = 1$
Show that the three planes Π_1, Π_2 and Π_3 do not intersect, and describe their geometrical configuration.

81 Find the angle between the line $\mathbf{r} = \begin{pmatrix} 1 \\ 6 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix}$ and the plane $3x - 4y + 2z = 10$.

82 Find the acute angle between the planes $2x + 3y - 5z = 4$ and $x - 2y + 4z = 9$.

4 Statistics and probability

Syllabus content

S4.1	**Sampling**		
	Book Section 6A	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Concepts of population, sample, random sample, discrete and continuous data.	Identify if data are continuous or discrete.	1	☐
	Identify in context what the population is, what the sample is and whether it is random.	2	☐
Reliability of data sources and bias in sampling.	Identify bias in sampling (a tendency for the sample to include more of one type of object).	3	☐
	Identify reliability of data (strictly the consistency of their results and in a more colloquial sense, how trustworthy they are).	4	☐
	Deal with missing data or errors in the recording of data.	5	☐
Interpretation of outliers.	Know that an outlier is defined as more than $1.5 \times \text{IQR}$ from the nearest quartile, and be able to suggest how to determine if an outlier should be removed from the sample.	6	☐
Sampling techniques and their effectiveness.	Be able to identify and evaluate the following sampling techniques: • simple random • convenience • systematic • quota • stratified.	7	☐
	Calculate the number of data items in each category of a stratified sample.	8	☐

S4.2	**Statistical diagrams**		
	Book Section 6C	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Presentation of data: Frequency distributions.	Interpret frequency distribution tables.	9	☐
Histograms.	Interpret frequency histograms.	10	☐
Cumulative frequency graphs.	Interpret cumulative frequency graphs, including finding median, quartiles, percentiles, range and interquartile range.	11	☐
Box and whisker plots.	Produce box and whisker diagrams.	12	☐
	Interpret box and whisker diagrams, including using them to compare distributions and use their symmetry to determine if a normal distribution is plausible.	13	☐

S4.3	Summary statistics		
	Book Section 6B	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Measures of central tendency.	Calculate the mean, median and mode of data.	14	☐
	Use the formula for the mean of data: $\bar{x} = \frac{\sum_{i=1}^{k} f_i x_i}{n}$ where $n = \sum_{i=1}^{k} f_i$.	15	☐
Estimation of mean from grouped data.	Use mid-interval values to estimate the mean of grouped data.	16	☐
Modal class.	Find the modal class for grouped data using tables or histograms.	17	☐
Measures of dispersion.	Use technology to calculate interquartile range (IQR), standard deviation and variance.	18	☐
Effect of constant changes on the original data.	Calculate the mean and standard deviation (and other statistics) of the new data set after a constant change.	19	☐
Quartiles of discrete data.	Use technology to obtain quartiles.	20	☐

S4.4	Correlation and regression		
	Book Section 6D	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Linear correlation of bivariate data: Pearson's product moment correlation coefficient, r.	Calculate the correlation coefficient of bivariate data using technology, and interpret the result, including being aware that correlation does not imply causation.	21	☐
Scatter diagrams.	Estimate the line of best fit by eye, knowing that it should pass through the mean point.	22	☐
Equation of the regression line of y on x.	Calculate the equation of the regression line using technology.	23	☐
Use of the equation of the regression line for prediction purposes.	Use the regression line while being aware of the dangers of extrapolation. Be aware of when a y-on-x regression line is appropriate.	24	☐
Interpret the meaning of the parameters, a and b, in a linear regression.	Put the meaning of the parameters into context.	25	☐
Piecewise linear models.	Create and use piecewise linear models.	26	☐

S4.5	Definitions in probability		
	Book Section 7A	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Concept of trial, outcome, equally likely outcomes, relative frequency, sample space and event.	Estimate probability from observed data.	27	☐
The probability of an event A is $P(A) = \frac{n(A)}{n(U)}$.	Find theoretical probabilities by listing all possibilities.	28	☐
The complementary events A and A'.	Link the probability of an event occurring and it not occurring. $P(A) + P(A') = 1$	29	☐
Expected number of occurrences.	Calculate how many times an outcome will be observed by multiplying the number of trials and the probability.	30	☐

S4.6	**Probability techniques**		
	Book Section 7B	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Use of Venn diagrams, tree diagrams, sample space diagrams and tables of outcomes to calculate probabilities.	Use Venn diagrams to organize information and find probabilities.	31	☐
	Use tree diagrams to organize information and find probabilities. In tree diagrams you multiply along the branches and add between the branches.	32	☐
	Use sample space diagrams to organize information and find probabilities.	33	☐
	Use tables of outcomes to organize information and find probabilities.	34	☐
Combined events.	Work with the notation $A \cap B$ meaning A and B occurring. Work with the notation $A \cup B$ meaning A or B or both occurring. Use: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$	35	☐
Mutually exclusive events.	Know that mutually exclusive means that the two events cannot both occur, so that $P(A \cap B) = 0$. Therefore $P(A \cup B) = P(A) + P(B)$.	36	☐
Conditional probability.	Know that $P(A\|B)$ means the probability of A given that B has happened. Use Venn diagrams, tree diagrams, sample space diagrams or tables of outcomes to find conditional probabilities.	37	☐
Independent events.	Know that if two events, A and B, are independent (that is, do not affect each other) then $P(A \cap B) = P(A)P(B)$	38	☐

S4.7	**Discrete random variables**		
	Book Section 8A	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Concept of discrete random variables and their distribution.	Create probability distributions from context.	39	☐
	Use the fact that the total probability in a probability distribution equals 1.	40	☐
Expected value (mean) for discrete data.	Use: $E(X) = \sum xP(x = x)$	41	☐
Applications.	Use probability distributions to answer questions in context.	42	☐
	Know that $E(X) = 0$ indicates a fair game if X represents the gain of a player.	43	☐

S4.8	**Binomial distribution**		
	Book Section 8B	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Binomial distribution.	Recognize that if a situation has • a fixed number of trials • outcomes that can be classified into two, 'successes' and 'failures' • fixed probability of being in each group • independent trials then the number of successes follows a binomial distribution.	44	☐
	Use technology to calculate binomial probabilities.	45	☐
Mean and variance of the binomial distribution.	Use: $E(X) = np$ $\text{Var}(X) = np(1-p)$ where X is the number of successes when there are n binomial trials each with a probability p of success.	46	☐

S4.9	Normal distribution		
	Book Section 8C	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
The normal distribution and curve; properties of the normal distribution.	Recognize that many natural situations are well modelled by a normal distribution. One way to validate this is to use the fact that about 68% of normally distributed data should fall within one standard deviation of the mean, about 95% within two standard deviations and about 99.7% within three standard deviations.	47	☐
Diagrammatic representation.	Recognize that a normal distribution can be represented by a symmetric bell-shaped curve with area representing probability.	48	☐
Normal probability calculations.	For a given mean and standard deviation, find the probability of a random variable falling in a given interval.	49	☐
Inverse normal calculations.	For a given probability, find the boundary of the region it describes.	50	☐

S4.10	***x*-on-*y* regression**		
	Book Section 19A	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Equation of the regression line of x on y.	Use your GDC to find the regression line of x on y for a given data set.	51	☐
Use of the equation for prediction purposes.	Use the regression line to predict values of x for given values of y.	52	☐

S4.11	**Formal conditional probability**		
	Book Section 19B	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Formal definition and use of the formulae: $P(A\|B) = \frac{P(A \cap B)}{P(B)}$ for conditional probabilities, $P(A\|B) = P(A)$ for independent events.	Find conditional probabilities using the formula $P(A\|B) = \frac{P(A \cap B)}{P(B)}$	53	☐
	Test whether two events are independent.	54	☐

S4.12	**Standardizing normal variables**		
	Book Section 19C	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Standardization of normal variables (z-values).	Find z-values and know that they give the number of standard deviations from the mean. Use the definition: $z = \frac{x - \mu}{\sigma}$	55	☐
Inverse normal calculations where mean and standard deviation are unknown.	Use the inverse normal distribution on your GDC and z-values to find an unknown mean and standard deviation.	56	☐

H4.13 Bayes' theorem

Book Section 9A | Revised ☐

Syllabus wording	*You need to be able to:*	*Question*
Use of Bayes' theorem for a maximum of three events.	Find the conditional probability $P(B\|A)$ when given $P(A\|B)$, either by using: $P(B\|A) = \frac{P(B)P(A\|B)}{P(B)P(A\|B) + P(B')P(A\|B')}$ or by drawing a tree diagram and using the conditional probability formula: $P(B\|A) = \frac{P(A \cap B)}{P(A)}$	57 ☐
	Extend to the case where there are three possible outcomes for event *B*. You can usually use the tree diagram method, but the formula is also given in the formula book: $P(B_i\|A) = \frac{P(B_i)P(A\|B_i)}{P(B_1)P(A\|B_1) + P(B_2)P(A\|B_2) + P(B_3)P(A\|B_3)}$	58 ☐

H4.14 Random variables

Book Section 9B, 9C | Revised ☐

Syllabus wording	*You need to be able to:*	*Question*
Variance of a discrete random variable.	Calculate the variance of a discrete random variable, using: $\mathrm{Var}(X) = \sum x^2 P(X = x) - \mu^2$ where $\mu = E(X)$.	59 ☐
Continuous random variables and their probability density function.	Know that a continuous random variable *X* can be represented by a probability density function f(*x*), which satisfies $0 \leqslant f(x) \leqslant 0$ and $\int_{-\infty}^{\infty} f(x)\,dx = 1$.	60 ☐
	Find the probability of a continuous random variable taking a value in a given interval: $P(a \leqslant X \leqslant b) = \int_a^b f(x)\,dx$.	61 ☐
	Work with pdfs defined piecewise.	62 ☐
Mode and median of continuous random variables.	Know that the mode corresponds to the maximum value of f(*x*).	63 ☐
	Find the median *m* by using $\int_{-\infty}^{m} f(x)dx = \frac{1}{2}$.	64 ☐
	For a piecewise defined pdf, identify in which part the median lies.	65 ☐
Mean, variance and standard deviation of both discrete and continuous random variables.	Understand the notation $E(X)$, $E(X^2)$ and $\mathrm{Var}(X)$, and know that $\mathrm{Var}(X) = E(X^2) - E(X)^2$. Remember that the standard deviation is the square root of the variance.	66 ☐
	Find the mean and variance of a continuous random variable by using $E(X) = \int x\, f(x)dx$ and $E(X^2) = \int x^2\, f(x)dx$.	67 ☐
	When finding mean and variance for a piecewise pdf, you need to split the integrals into two parts.	68 ☐
	Use $E(X)$ to determine whether a game is fair, or to find the cost needed to make a game fair.	69 ☐
The effect of linear transformations of *X*.	Use $E(aX + b) = aE(X) + b$ and $\mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X)$.	70 ☐

Practice questions

1 Determine whether each of the following variables are continuous or discrete.

a Number of people in a family.

b Time for a nucleus to decay.

c Age in complete years.

2 A doctor wants to find out about whether exercise can lower the incidence of illness. He asks patients who come to his clinic to fill in a survey about their exercise habits. 20% of them agree to do this.

a Suggest a possible population that the doctor is interested in.

b Is his sample random?

3 Is the sampling in question **2** likely to be biased? Justify your answer.

4 Five independent groups of people were asked to estimate the length of an arrow which is 5 cm long. The average for the groups was 4.6 cm, 4.6 cm, 4.7 cm, 4.8 cm, 4.8 cm. Does this suggest that the results are reliable?

5 Five people were asked to record their height in metres:
A: 1.83 B: 1.45 C: 1.77 D: 5.10 E: 1.60
Suggest which data item is an error. What should be done with this item?

6 A data set has lower quartile 7 and upper quartile 11. Explain why 18 should be considered an outlier and suggest how to determine if it should be excluded from the data.

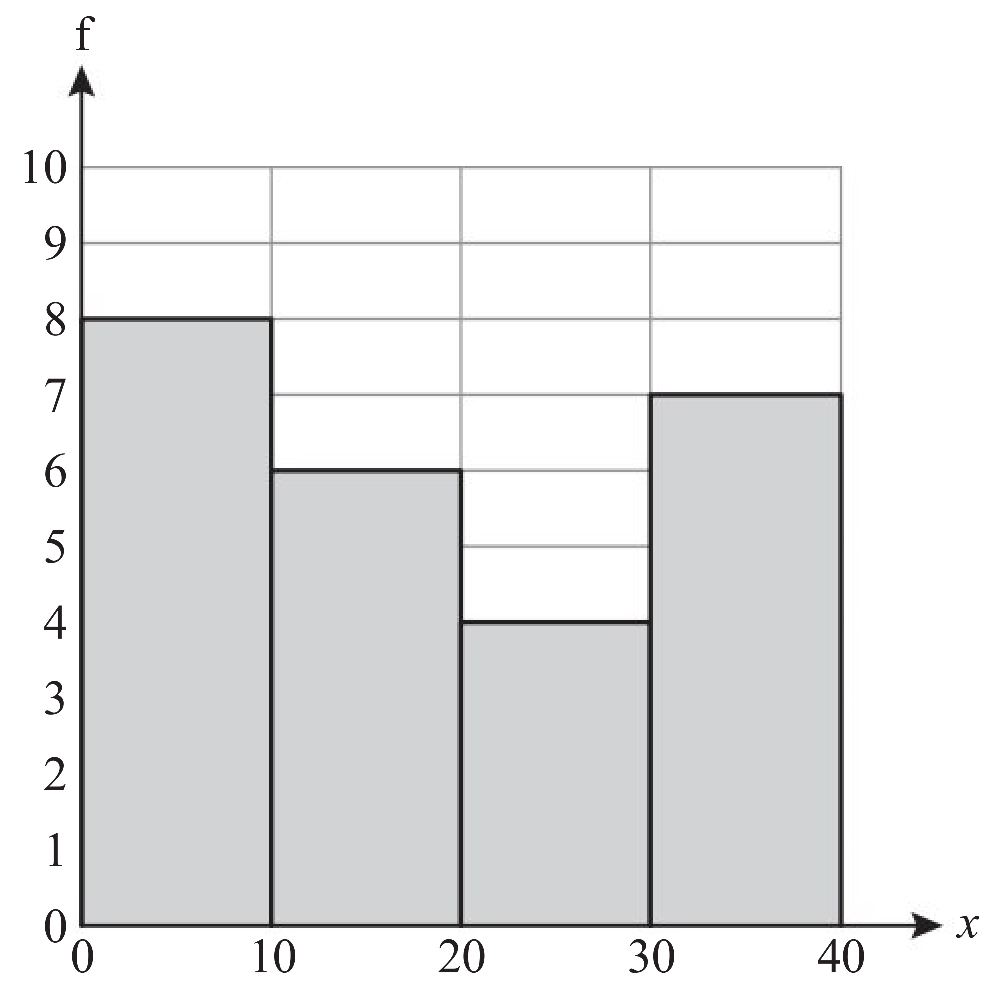
7 Write down the sampling method used by the doctor in question **2**.

8 A language school consists of students from either Italy or Spain. There are 60 from Italy and 90 from Spain. In a stratified sample of 20 students, how many should be from Italy?

9 For the following frequency table, find the proportion of data items above 20.

x	$0 < x \leqslant 20$	$20 < x \leqslant 30$	$30 < x \leqslant 40$
Frequency	15	18	12

10 For the following histogram, estimate the number of data items above 25.

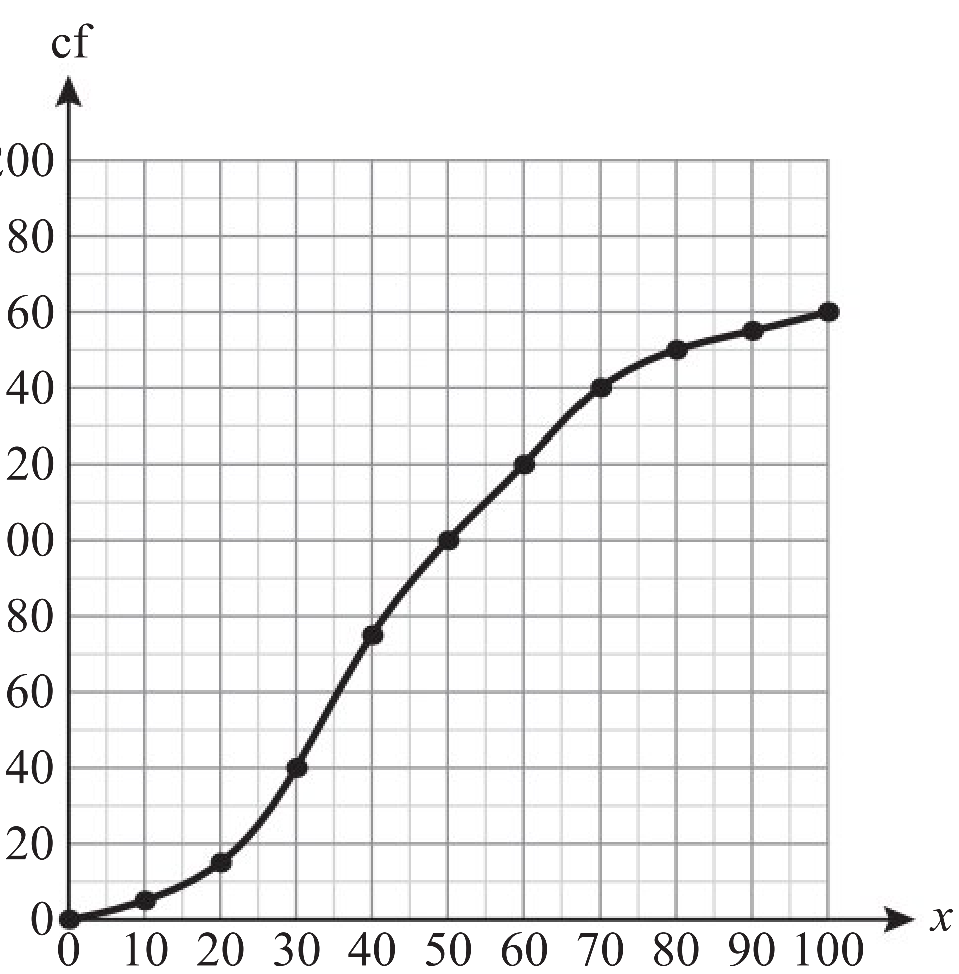


11 For the following cumulative frequency graph find

a the median

b the interquartile range

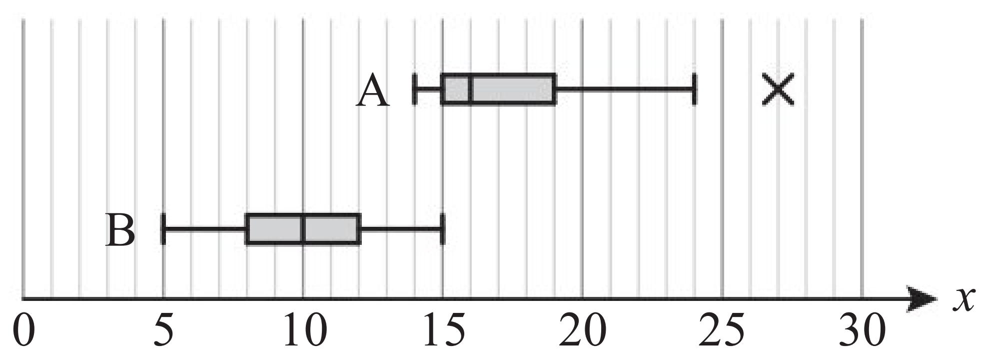
c the 90th percentile.



12 Sketch a box and whisker plot for the sample below:
12, 13, 15, 16, 16, 18, 18, 19, 20.

13 For the following box and whisker plots:
a Compare the two distributions.

b Determine, with justification, which of the two distributions is more likely to be a normal distribution.



14 Write down the mean, median and mode of the following data:
14, 14, 16, 17, 19, 20, 23, 25.

15 The numbers 4, 8, 2, 9 and x have a mean of 7.
Find the value of x.

16 **a** Estimate the mean of the following grouped data.

x	$10 < x \leqslant 20$	$20 < x \leqslant 30$	$30 < x \leqslant 50$	$50 < x \leqslant 60$
Frequency	10	12	15	13

b Explain why it is only an estimate.

17 Find the modal class for the data below:

x	$0 < x \leqslant 5$	$5 < x \leqslant 10$	$10 < x \leqslant 15$	$15 < x \leqslant 20$
Frequency	16	12	15	18

18 For the data set 6, 7, 9, 12, 14, 18, 22 find

a the interquartile range

b the standard deviation

c the variance.

19 A set of data has mean 12 and standard deviation 10. Every item in the data set is doubled, then 4 is added on. Find the mean and standard deviation of the new data set.

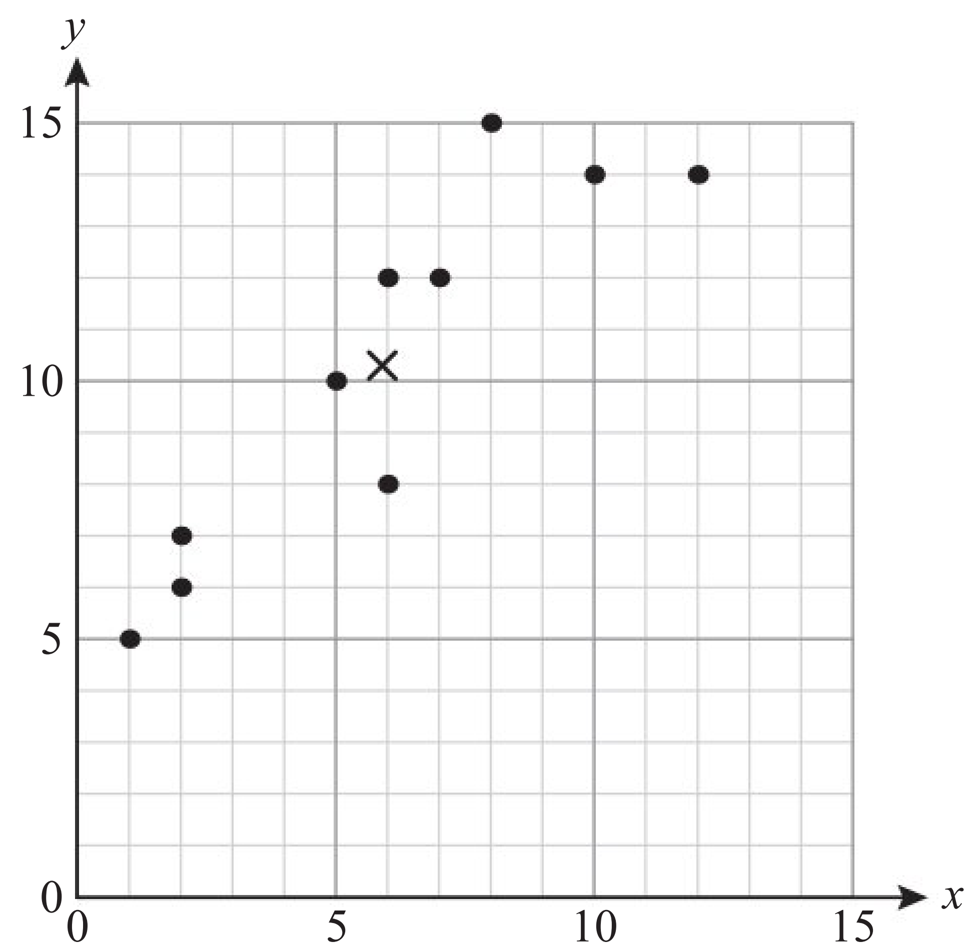
20 Find the quartiles of the following data:
17, 15, 23, 29, 15, 22, 28, 30.

21 a Calculate Pearson's product-moment correlation coefficient for the following data:

x	2	4	4	7
y	1	3	6	8

b Interpret your result.

22 The following diagram shows a set of 10 data items with the mean point labelled with a cross.



a Sketch a line of best fit on the diagram.
b Hence estimate y when $x = 3$.

23 Find the y-on-x regression line for the following data, in which y is dependent on x.

x	1	2	2	5	6	6	7	8	10	12
y	5	6	7	10	12	8	12	15	14	14

24 **a** Use your answer to question **23** to estimate:
i y when $x = 9$

ii y when $x = 20$

iii x when $y = 10$.

b Which of the predictions made in part **a** is valid. Justify your answer.

25 A social scientist investigates how the number of text messages sent by pupils each day (y) depends on the number of hours they spend on social media each day (x). He finds the regression line $y = 6.7 + 1.4x$. Interpret what each of the following numbers mean in context.
a 6.7

b 1.4

26 A veterinary researcher believes that the growth of a breed of snake is very different during their first six months compared to their next six months.
She collects the following data showing the length (L cm) and age (A months) of a sample of snakes.

A	1	2	4	4	7	7	10	11	12
L	4	8	15	18	30	32	34	36	34

a Create a piecewise linear model to reflect the researcher's belief.

b Use your answer to part **a** to estimate the length of a 3-month-old snake of this breed.

27 A coin is flipped 200 times and 134 heads are observed. Estimate the probability of observing a head when the coin is flipped.

28 Find the probability of rolling a prime number on a fair six-sided dice.

29 If $P(A) = 0.6$, find $P(A')$.

30 If there are 30 pupils in a class and the probability of a student being absent is 0.05, find the expected number of absent pupils.

31 In a class of 30 students, 14 study French, 18 study Spanish and 4 study both languages. Find the probability that a randomly chosen student studies neither French nor Spanish.

32 A drawer contains three white socks and five black socks. Two socks are drawn without replacement.

a Find P(2nd sock is black|1st sock is white).

b Find the probability that the socks are different colours.

33 A fair four-sided dice is thrown twice.

a What is the probability that the total score is greater than 5?

b If the total score is greater than 5, what is the probability that it is 7?

34 100 students were asked whether they preferred soccer or cricket. They were also asked if they prefer mathematics or art. The results are summarized below:

	Soccer	**Cricket**
Mathematics	40	20
Art	30	x

a Find the value of x.

b Find the probability that a randomly chosen student prefers mathematics to art.

35 If $P(A) = 0.5$, $P(B) = 0.7$ and $P(A \cap B) = 0.3$ find $P(A \cup B)$.

36 Events A and B are mutually exclusive. If $P(A) = 0.4$ and $P(B) = 0.2$, find $P(A \cup B)$.

37 For the sample in question **34**, determine the probability that a randomly chosen person who prefers soccer also prefers mathematics.

38 Independent events A and B are such that $P(A) = 0.4$ and $P(B) = 0.6$. Find $P(A \cap B)$.

39 A drawer contains three white socks and four black socks. Two socks are drawn at random without replacement. Find the probability distribution of W, the number of white socks drawn.

40 The random variable X can take values 0, 1 or 2 with probability $P(X = x) = k(x + 1)$.
Find the value of k.

41 For the distribution given below, find E(*X*).

x	0.5	1	2.5
P(*X* = *x*)	0.5	0.4	0.1

42 The value of prizes ($*X*) won by an individual each month in a prize draw is shown below.

X	0	10	2000
P(*X* = *x*)	0.9	0.095	0.005

a Given that an individual wins a prize, find the probability that it is $2000.

b Find the probability of winning more than the expected amount.

43 The gain, $*X*, of a player in a game of chance follows the distribution shown in the table.

X	–1	0	k
P(*X* = *x*)	0.6	0.3	0.1

Find the value of k that would make the game fair.

44 A drawer contains 5 black socks and 10 red socks. Four socks are drawn at random without replacement. Explain why the number of black socks drawn does not follow a binomial distribution.

45 If X is a random variable following a binomial distribution with five trials and a probability of success of 0.4, find

a $P(X = 2)$

b $P(X \geqslant 3)$.

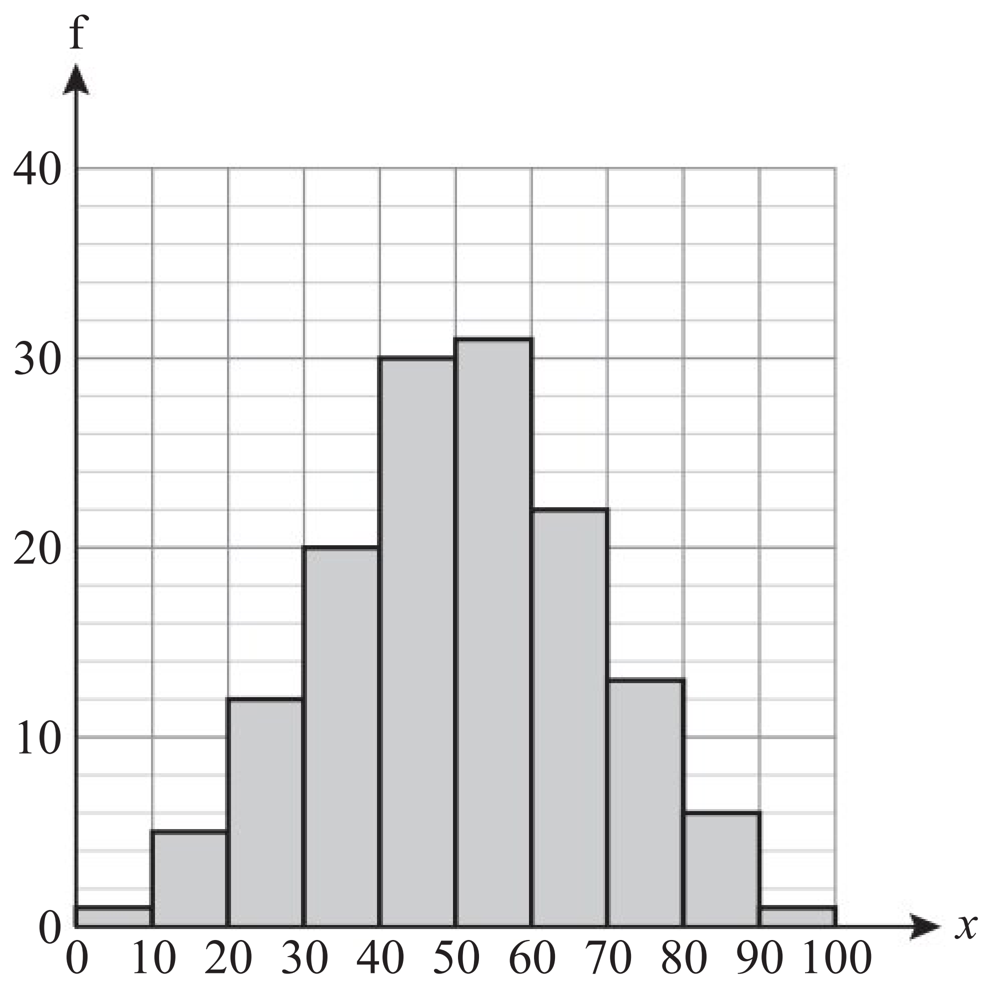
46 A biased coin has a probability of 0.6 of showing a head. It is tossed 10 times.
If this experiment is repeated many times:

a Find the expected mean number of heads.

b Find the expected standard deviation in the number of heads.

47 The time for a child to learn a new dance move is found to have a mean of 2 weeks and a standard deviation of 4 weeks.
Explain why this variable is unlikely to be modelled by a normal distribution.

48 The following histogram shows the results of an experiment.



a What feature of this graph suggests a normal distribution might be a good model for the outcome of the experiment?

b Visually estimate the mean of the distribution.

49 A random normal variable has mean 12 and standard deviation 2.
Find the probability that an observation is between 11 and 15.

50 A random normal variable has mean 100 and standard deviation 15. The probability of being above k is 0.7.
Find the value of k.

51 Find the equation of the regression line of x on y for the following data:

x	1	2	3	3	4	5	7	8
y	18	20	17	15	16	12	10	11

52 Based on data with y-values between 15 and 30 and a correlation coefficient of 0.89, the following regression line is found:

$$x = 1.82y - 11.5$$

a Use this line to predict the value of x when

i $y = 20$

ii $y = 35$.

b Comment on the reliability of your predictions in each case.

53 $P(A) = 0.6$ and $P(A \cap B) = 0.4$.
Find $P(B|A)$.

54 $P(A) = 0.2$, $P(B) = 0.8$ and $P(A \cap B) = 0.7$.
Determine whether A and B are independent.

55 Given that $X \sim N(10, 4.8^2)$, find how many standard deviations $x = 17$ is from the mean.

56 $X \sim N(\mu, \sigma^2)$ with $P(X < 12) = 0.3$ and $P(X > 34) = 0.2$.
Find μ and σ.

57 You are given the following information about events A and B:
$P(B) = 0.6$, $P(A|B) = 0.4$, $P(A|B') = 0.8$.
Find

a $P(B|A)$

b $P(B|A')$.

58 Renzhi takes a lift to school in the morning with either Aline (20% of the time), Brett (35% of the time) or Carlos (45% of the time). The probability of being late in each case is: 0.05 with Aline, 0.16 with Brett and 0.02 with Carlos.
Given that Renzhi is late for school, what is the probability that he took a lift with Aline?

59 Find the mean and variance of the discrete random variable X with the following probability distribution:

x	0	1	2
$P(X = x)$	k	$2k$	$4k$

60 Show that the function

$$f(x) = \begin{cases} \frac{k}{2}\sin(kx) & \text{for } 0 \leqslant x \leqslant \frac{\pi}{k} \\ 0 & \text{otherwise} \end{cases}$$

is a probability density function for all $k > 0$.

61 The random variable X has the probability density function given by

$$f(x) = \begin{cases} k\sqrt{x} & \text{for } 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $P\left(X > \frac{1}{4}\right)$.

62 The probability density function of a random variable Y is given by

$$g(y) = \begin{cases} \dfrac{4\pi}{4+\pi}\sin(\pi y) \text{ for } 0 \leqslant y < \dfrac{1}{2} \\ \dfrac{4\pi}{4+\pi}(2-2y) \text{ for } \dfrac{1}{2} \leqslant y \leqslant 1 \\ 0 \text{ otherwise} \end{cases}$$

Find $P\left(\frac{1}{4} < Y \leqslant \frac{3}{4}\right)$.

63 Two random variables, X and Y, have the probability density functions given below.

Find the mode of X and the mode of Y.

$$f(x) = \begin{cases} \dfrac{3}{4}x(x-2)^2 \text{ for } 0 \leqslant x < 2 \\ 0 \text{ otherwise} \end{cases} \qquad g(y) = \begin{cases} \dfrac{6}{19}y^2 - \dfrac{18}{19}y + \dfrac{39}{38} \text{ for } 0 \leqslant y \leqslant 2 \\ 0 \text{ otherwise} \end{cases}$$

64 Find the median of the random variable with the probability density function

$f(x) = \dfrac{\ln x}{\ln 64 - 2}$ for $2 \leqslant x \leqslant 4$ and $f(x) = 0$ otherwise.

65 A continuous random variable X has the probability density function given by

$$f(x) = \begin{cases} kx \text{ for } 0 \leqslant x < 1 \\ k \text{ for } 1 \leqslant x \leqslant 2 \\ 0 \text{ otherwise} \end{cases}$$

Find the median of X.

66 The discrete random variable X has the probability distribution given by $P(X = x) = \frac{12}{13x}$ for $x = 2, 3, 4$.

a Find $E(X)$ and $E(X^2)$.

b Hence find the standard deviation of X.

67 Continuous random variable X takes all real values between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, and has the probability density function $f(x) = \frac{1}{2}\cos x$. Find the variance of X.

68 Find the standard deviation of the random variable X with the probability density function

$$f(x) = \begin{cases} \frac{x}{6} \text{ for } 0 \leqslant x \leqslant 2 \\ \frac{1}{2} - \frac{x}{12} \text{ for } 2 < x \leqslant 6 \\ 0 \text{ otherwise} \end{cases}$$

69 In a game, a player needs to throw a ball as far as possible. They win the amount of money (in Yen) equal to the length of the throw, in metres. Let X be the continuous random variable modelling the length of a throw. It is assumed that no one can throw further than 10 m, and that the probability density function of X is

$$f(x) = \frac{0.328e^x}{e^x + e^7} \text{ for } 0 \leqslant x \leqslant 10$$

a How much should be charged per game in order to make the game fair?

b What percentage of players are expected to make a profit from this game?

70 Random variable X has mean 12 and variance 24. Find the mean and variance of $80 - 3X$.

5 Calculus

Syllabus content

S5.1	The concepts of a limit and derivative		
	Book Section 9A	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Introduction to the concept of a limit.	Estimate the value of a limit from a table.	1	☐
	Estimate the value of a limit from a graph.	2	☐
Derivative interpreted as gradient function and as rate of change.	Understand and use the notation for derivatives: $\frac{dy}{dx}$ and $f'(x)$.	3	☐
	Interpret the derivative as a rate of change.	4	☐
	Interpret the derivative as a gradient function.	5	☐
	Estimate the gradient at a point as a limit of gradients of chords.	6	☐

S5.2	Increasing and decreasing functions		
	Book Section 9B	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Graphical interpretation of $f'(x) > 0$, $f'(x) = 0$, $f'(x) < 0$.	Identify intervals on which a function is increasing ($f'(x) > 0$) and decreasing ($f'(x) < 0$).	7	☐
	Sketch the graph of the derivative from the graph of a function.	8	☐
	Sketch the graph of a function from the graph of its derivative.	9	☐

S5.3	Derivatives of polynomials		
	Book Section 9C	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
The derivative of functions of the form $f(x) = ax^n + bx^{n-1} + \cdots$ where all exponents are integers.	Apply the rule to differentiate polynomials using $f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$	10	☐
	Rearrange an expression into the form $f(x) = ax^n + bx^{n-1} + \cdots$ before differentiating.	11	☐

S5.4	Equations of tangents and normals		
	Book Section 9D	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Tangents and normals at a given point, and their equations.	Evaluate the gradient at a given point.	12	☐
	Find the point of the curve with a given gradient.	13	☐
	Find the equation of the tangent to the curve $y = f(x)$ at the point (x_1, y_1) using $y - y_1 = m(x - x_1)$ where $y_1 = f(x_1)$ and $m = f'(x_1)$.	14	☐
	Find the equation of the normal to the curve using $y - y_1 = -\frac{1}{m}(x - x_1)$	15	☐
	Solve problems involving tangents and normals.	16	☐
	Use technology to find the gradient and the equation of the tangent at a given point.	17	☐
	Use technology to draw the graph of the gradient function.	18	☐

S5.5	Introduction to integration		
	Book Section 10A, 10B	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Integration as anti-differentiation of functions of the form $f(x) = ax^n + bx^{n-1} + \cdots$, where $n \in \mathbb{Z}, n \neq -1$.	Use $\int ax^n \mathrm{d}x = \frac{a}{n+1}x^{n+1} + c$, for $n \neq -1$.	19	☐
	Rearrange an expression into the form $f(x) = ax^n + bx^{n-1} + \ldots$ before integrating.	20	☐
Definite integrals using technology. Area of a region enclosed by a curve $y = f(x)$ and the x-axis, where $f(x) > 0$.	Use technology to evaluate integrals of the form $\int_a^b f(x)\,\mathrm{d}x$, and interpret this as the area between the curve and the x-axis.	21	☐
Anti-differentiation with a boundary condition to determine the constant term.	Find the expression for y in terms of x when given $\frac{\mathrm{d}y}{\mathrm{d}x}$ and one pair of (x, y) values.	22	☐

S5.6	Further differentiation		
	Book Section 20A, 20B, 20C	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Derivative of x^n ($n \in \mathbb{Q}$), $\sin x$, $\cos x$, e^x and $\ln x$. Differentiation of a sum and a multiple of these functions.	Apply the rules of differentiation to these functions.	23	☐
	Evaluate the gradient at a given point.	24	☐
The chain rule for composite functions.	$y = g(u)$ where $u = f(x) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$	25	☐
The product and quotient rules.	$y = uv \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$	26	☐
	$y = \frac{u}{v} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$	27	☐

S5.7	Second derivative		
	Book Section 20D	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
The second derivative.	Find the second derivative and understand the notation $f''(x)$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.	28	☐
Graphical behaviour of functions, including the relationship between the graphs of f, f′ and f″.	Sketch the graph of $y = f''(x)$ given the graph of $y = f(x)$.	29	☐
	Describe sections of a graph as 'concave up' or 'concave down'.	30	☐
	Use the second derivative to determine whether a graph is concave up ($f''(x) > 0$) or concave down ($f''(x) < 0$).	31	☐

S5.8	Maximum, minimum and inflection points		
	Book Section 20E, 20F	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Local maximum and minimum points. Testing for maximum and minimum.	Locate local maximum and minimum points by solving $f'(x) = 0$.	32	☐
	Use the second derivative to distinguish between maximum ($f''(x) < 0$) and minimum ($f''(x) > 0$).	33	☐
Optimization.	Find the maximum or minimum value of a function in a real-life context.	34	☐
Points of inflection with zero and non-zero gradients.	Locate points of inflection by solving $f''(x) = 0$ and checking that the concavity of the function changes at that point.	35	☐

S5.9	Book Section 21C	**Introduction to kinematics** Revised ☐	
Syllabus wording		*You need to be able to:*	*Question*
Kinematic problems involving displacement s, velocity v, acceleration a and total distance travelled.		Use differentiation to find velocity and acceleration. $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$	36 ☐
		Find speed as the magnitude of velocity.	37 ☐
		Integrate velocity to find the displacement. $\text{displacement} = \int_{t_1}^{t_2} v(t)\,dt$	38 ☐
		Integrate speed to find distance travelled. $\text{distance} = \int_{t_1}^{t_2} \lvert v(t)\rvert\,dt$	39 ☐

S5.10	Book Section 21A	**Further integration techniques** Revised ☐	
Syllabus wording		*You need to be able to:*	*Question*
Indefinite integral of x^n $(n \in \mathbb{Q})$, $\sin x$, $\cos x$, $\frac{1}{x}$ and e^x.		Apply integration rules to these functions. $\int \frac{1}{x}\,dx = \ln\lvert x\rvert + c$ $\int \sin x\,dx = -\cos x + c$ $\int \cos x\,dx = \sin x + c$ $\int e^x\,dx = e^x + c$	40 ☐
The composites of any of these with the linear function $ax + b$.		When integrating $f(ax + b)$, remember to divide by a.	41 ☐
Integration by inspection (reverse chain rule) or by substitution for expressions of the form: $\int kg'(x)f(g(x))\,dx$		Use reverse chain rule for integrals where a composite function is multiplied by the derivative of the inner function: $\int g'(x)f(g(x))\,dx = f(g(x)) + c$	42 ☐

S5.11	Book Section 21B	**Evaluating definite integrals** Revised ☐	
Syllabus wording		*You need to be able to:*	*Question*
Definite integrals, including the analytical approach.		Evaluate definite integrals by using $\int_a^b g'(x)\,dx = g(b) - g(a)$	43 ☐
		Recognize that some definite integrals can only be found using technology.	44 ☐
Areas of a region enclosed by a curve $y = f(x)$ and the x-axis, where $f(x)$ can be positive or negative, without the use of technology.		Use definite integrals to find areas without a calculator using $A = \int_a^b \lvert y\rvert\,dx$	45 ☐
		Split the area into positive and negative parts before calculating the definite integrals.	46 ☐
Areas between curves.		Find the area between two curves by using $\int_a^b (f_1(x) - f_2(x))\,dx$ You may need to find the intersection points first.	47 ☐

H5.12 Continuity and differentiability

Book Section 10A — Revised ☐

Syllabus wording	You need to be able to:	Question
Informal understanding of continuity and differentiability of a function at a point.	Determine whether a function is continuous by considering its value on either side of a point.	48 ☐
	Determine whether a function is differentiable by considering the value of its derivative on either side of a point.	49 ☐
Understanding of limits (convergence and divergence).	Evaluate a limit of a function or show that a function diverges to infinity as x tends to a given value.	50 ☐
Definition of the derivatives from first principles.	Find the derivative of a polynomial by using $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.	51 ☐
Higher derivatives.	Understand the notation $\frac{d^n y}{dx^n}$ and $f^{(n)}(x)$.	52 ☐

H5.13 L'Hôpital's rule and evaluation of limits

Book Section 10B, 10D — Revised ☐

Syllabus wording	You need to be able to:	Question
The evaluation of limits of the form $\lim_{x \to a} \frac{f(x)}{g(x)}$ and $\lim_{x \to \infty} \frac{f(x)}{g(x)}$ using l'Hôpital's rule or the Maclaurin series.	Evaluate limits of the form $\frac{0}{0}$ and $\frac{\infty}{\infty}$ using l' Hôpital's rule: $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$	53 ☐
	Evaluate limits of the form $\frac{0}{0}$ by replacing top and bottom by their Maclaurin series.	54 ☐
Repeated use of l'Hôpital's rule.	Use the rule repeatedly until the limit can be evaluated.	55 ☐

H5.14 Applications of differentiation

Book Section 10C, 10D, 10E — Revised ☐

Syllabus wording	You need to be able to:	Question
Implicit differentiation.	Use implicit differentiation, in combination with chain, product and quotient rules, to find $\frac{dy}{dx}$.	56 ☐
Related rates of change.	Use the chain rule to find related rates of change.	57 ☐
Optimization problems.	Find maximum and minimum values of a function in practical problems, remembering that they may occur at the end of the interval.	58 ☐

H5.15 Further derivatives and integrals

Book Section 10F — Revised ☐

Syllabus wording	You need to be able to:	Question
Derivatives of $\tan x$, $\sec x$, $\csc x$, $\cot x$, a^x, $\log_a x$, $\arcsin x$, $\arccos x$, $\arctan x$.	Use these standard derivatives in combination with chain, product and quotient rules.	59 ☐
Indefinite integrals of the derivatives of any of the above functions.	Use these standard integrals.	60 ☐
The composites of any of these with a linear function.	Remember to divide by a when integrating $f(ax + b)$.	61 ☐
Use of partial fractions to rearrange the integrand.	Split a fraction with a quadratic denominator into partial fractions before integrating.	62 ☐

H5.16	**Advanced integration techniques**		
	Book Section 10G, 10H	Revised ☐	
Syllabus wording	*You need to be able to:*		*Question*
Integration by substitution.	Use a given substitution to find an integral		63 ☐
	With definite integrals, change limits before integrating.		64 ☐
Integration by parts.	Use the integration by parts formula: $\int u \frac{dv}{dx}\,dx = uv - \int v \frac{du}{dx}\,dx$ Be able to choose u and v.		65 ☐
Repeated integration by parts.	Use integration by parts twice (this is usually needed with integrals such as $\int x^2 e^x\,dx$ and $\int x^2 \cos x\,dx$).		66 ☐

H5.17	**Further areas and volumes**		
	Book Section 10I	Revised ☐	
Syllabus wording	*You need to be able to:*		*Question*
Area of the region enclosed by a curve and the y-axis.	Use $\int_c^d \lvert x\rvert\,dy$; you may need to express x in terms of y first.		67 ☐
Volumes of revolution about the x-axis or y-axis.	Use $\int_a^b \pi y^2\,dx$ to find a volume of revolution.		68 ☐
	Use $\int_c^d \pi x^2\,dy$; you may need to express x in terms of y first.		69 ☐

H5.18	**Differential equations**		
	Book Section 11A, 11B, 11C	Revised ☐	
Syllabus wording	*You need to be able to:*		*Question*
First order differential equations.	Form differential equations, understanding that the derivative represents the rate of change.		70 ☐
Numerical solution of $\frac{dy}{dx} = f(x, y)$ using Euler's method.	For a given step size h, use $x_{n+1} = x_n + h,\ y_{n+1} = y_n + hf(x_n, y_n)$.		71 ☐
Variables separable.	Separate variables and integrate both sides. You may need to factorize first. The general solution will have one unknown constant in the final answer.		72 ☐
	Use the initial condition to find the constant of integration. The question will tell you whether you need to rearrange your final answer into the form $y = f(x)$.		73 ☐
Homogeneous differential equation $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ using the substitution $y = vx$.	Use the substitution $y = vx$ (so $\frac{dy}{dx} = v + x\frac{dv}{dx}$) to transform the equation into one that can be solved using either separation of variables or the integrating factor.		74 ☐
Solution of $y' + P(x)y = Q(x)$ using the integrating factor.	Use the integrating factor $I = e^{\int P(x)dx}$ and then $Iy = \int IQ\,dx$.		75 ☐

H5.19	**Maclaurin series**		
	Book Section 11D, 11E	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Maclaurin series to obtain expressions for e^x, $\sin x$, $\cos x$, $\ln(1+x)$, $(1+x)^p$, $p \in \mathbb{Q}$.	Derive Maclaurin series by using $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots$	76	☐
Use of simple substitution, products, integration and differentiation to obtain other series.	Derive Maclaurin series of composite functions by substituting into the basic Maclaurin series.	77	☐
	Multiply two Maclaurin series.	78	☐
	Differentiate or integrate a Maclaurin series term-by-term.	79	☐
Maclaurin series developed from differential equations.	Write $y = \sum_{k=0}^{\infty} a_k x^k$, substitute into the differential equation and compare coefficients to find the general expression for the kth term of the Maclaurin series for y. Remember that $y' = \sum_{k=0}^{\infty}(k+1)a_{k+1}x^k$ and $y' = \sum_{k=0}^{\infty}(k+1)(k+2)a_{k+2}x^k$.	80	☐
	Differentiate the differential equation and use the values of $y(0)$, $y'(0)$, $y''(0)$, etc. to find the first few terms of the Maclaurin expansion.	81	☐

Practice questions

1 In this question, x is measured in degrees. Use a table to estimate, to two decimal places, the limit of $\frac{\sin 3x}{0.2x}$ when x tends to zero.

2 Use a graph to estimate the limit of $\frac{\ln\left(\frac{x}{2}\right)}{x-2}$ when x tends to 2.

3 Given that $y = 3x^2 - 5x$ and $\frac{dy}{dx} = 6x - 5$, what is the value of the derivative of y when $x = 2$?

4 Write an equation to represent the following situation:
The area decreases with time at a rate proportional to the current area.

5 The table shows some information about a function $f(x)$.

x	1	3	4
$f(x)$	4	8	5
$f'(x)$	−1	4	2

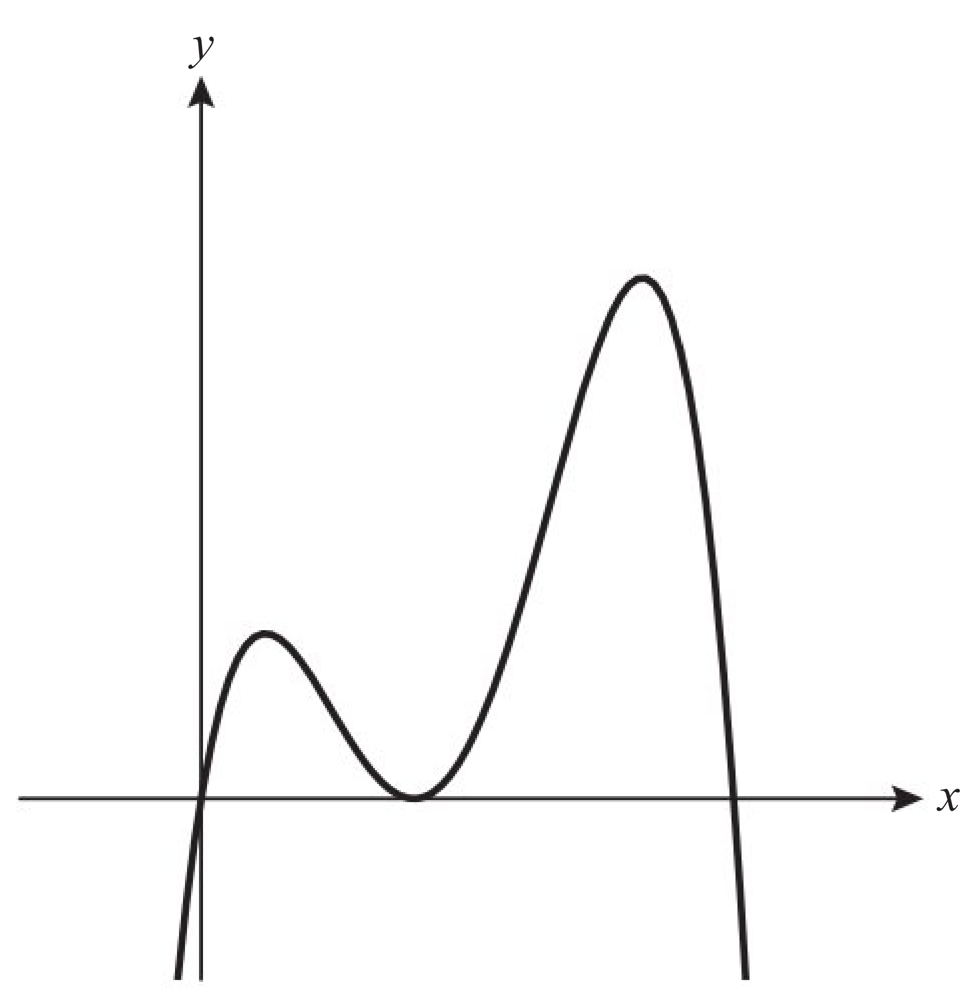
A graph has equation $y = f(x)$. Find the gradient of the graph at the point where $y = 4$.

6 Point $P(4, 2)$ lies on the curve with equation $y = \sqrt{x}$. The table shows the coordinates of a variable point Q and the gradient of the chord PQ.
Complete the table and use it to estimate the gradient of the curve at P.

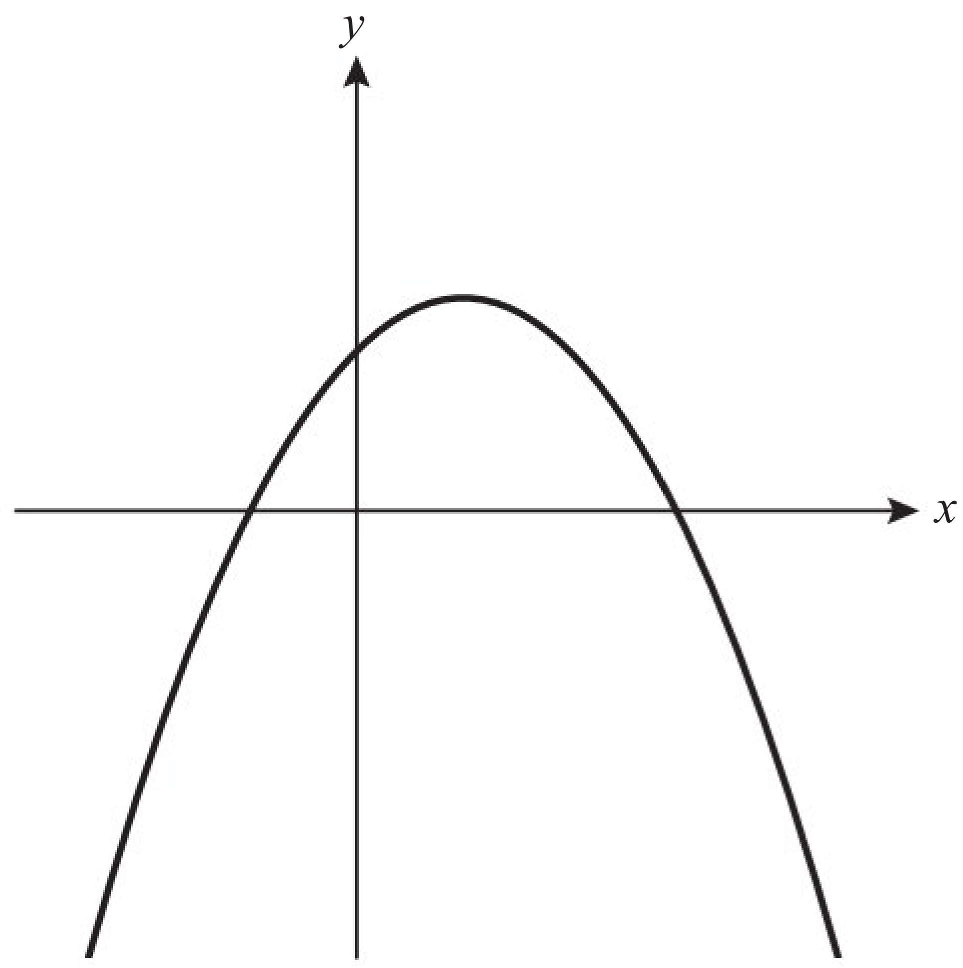
x_Q	y_Q	Δx	Δy	Gradient of PQ
5	2.236	1	0.236	0.236
4.1	2.025	0.1		
4.01				
4.001				

7 Use technology to sketch the graph of $f(x) = x^3 - 5x + 2$ and to find the range of values of x for which $f'(x) < 0$.

8 The graph of $y = f(x)$ is shown below. Sketch the graph of $y = f'(x)$.



9 The graph of $y = f'(x)$ is shown below. Sketch one possible graph of $y = f(x)$.



10 Differentiate $y = 4x^2 - \frac{1}{10}x^{-5} - 3x + 2$.

11 Find $f'(x)$ when

a $f(x) = 3x^2(4 - x^4)$

b $f(x) = 1 - \frac{3}{2x^4}$

c $f(x) = \frac{4x^2 - 3x + 1}{5x}$.

12 Given that $f(x) = 4x^2 - 2x^{-1}$, evaluate $f'(2)$.

13 Find the x-coordinates of the points on the curve $y = 12x + 5x^{-1}$ where the gradient equals 2.

14 A curve has equation $y = x^2 - 3$.
Find the equation of the tangent to the curve at the point where $x = 4$.

15 Find the equation of the normal to the curve $y = 3x - 2x^{-1}$ at the point where $x = 2$.

16 The tangent to the curve with equation $y = x^2 - 3$ at the point (a, b) passes through $(0, -12)$.
Find the possible values of a.

17 A curve has equation $y = \dfrac{4\sqrt{x} - 3}{7x^2}$.

Find, correct to two decimal places:

a the gradient when $x = 3.2$

b the equation of the tangent at the point where $x = 3.2$.

18 A curve has equation $y = \dfrac{4\sqrt{x} - 3}{7x^2}$.

Find the coordinates of the point on the curve where the gradient is 2.

19 Find $\int 9x^2 + 6x^{-3}\,dx$.

20 Find $\int \frac{x^5 - 3}{2x^2}\,dx$.

21 Given that $\frac{dy}{dx} = 4x + 2$, and that $y = 3$ when $x = 2$, find an expression for y in terms of x.

22 Find the area enclosed by the curve $y = 2x^3 - 1$, the x-axis and the lines $x = 2$ and $x = 3$.

23 Differentiate $3\sin x - 5\cos x + 2$.

24 Given that $f(x) = 2\sqrt{x} - 3 \ln x$, evaluate $f'(9)$.

25 Differentiate:

a $\sqrt{3x^2 - 1}$

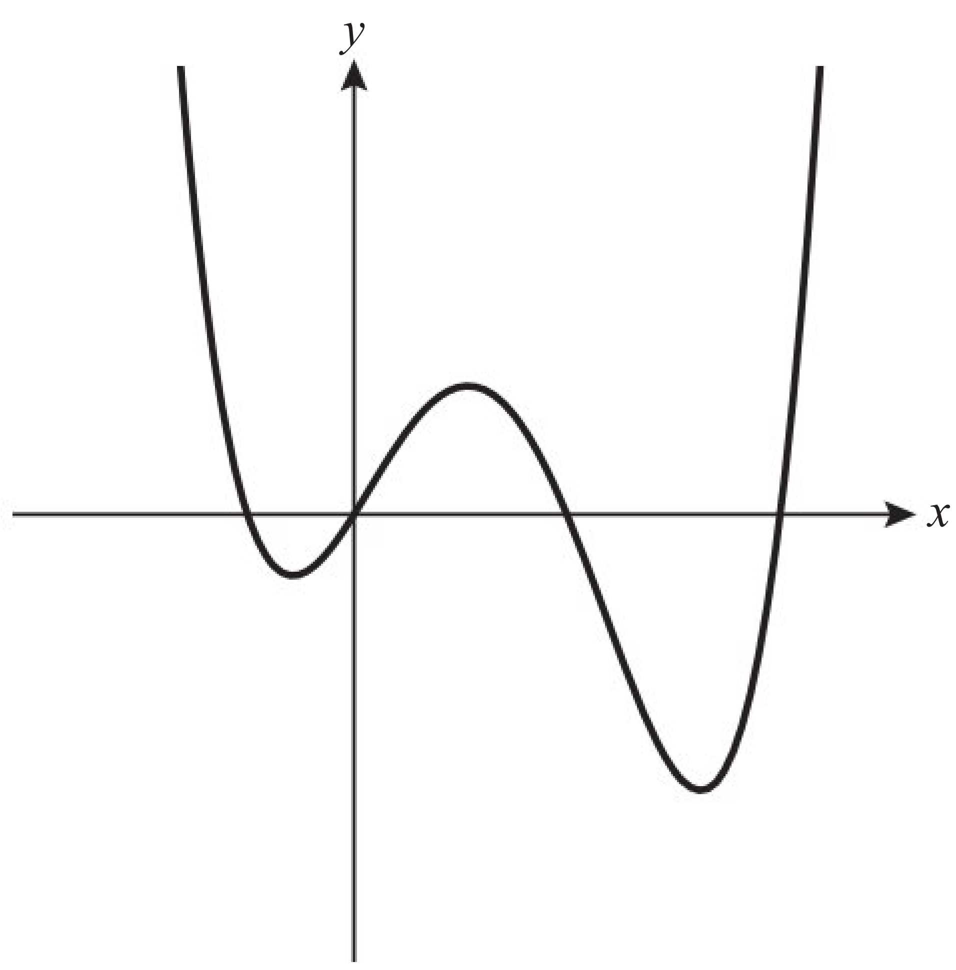
b $2\sin^3(5x)$.

26 Given that $y = 4xe^{-3x}$, find $\frac{dy}{dx}$.

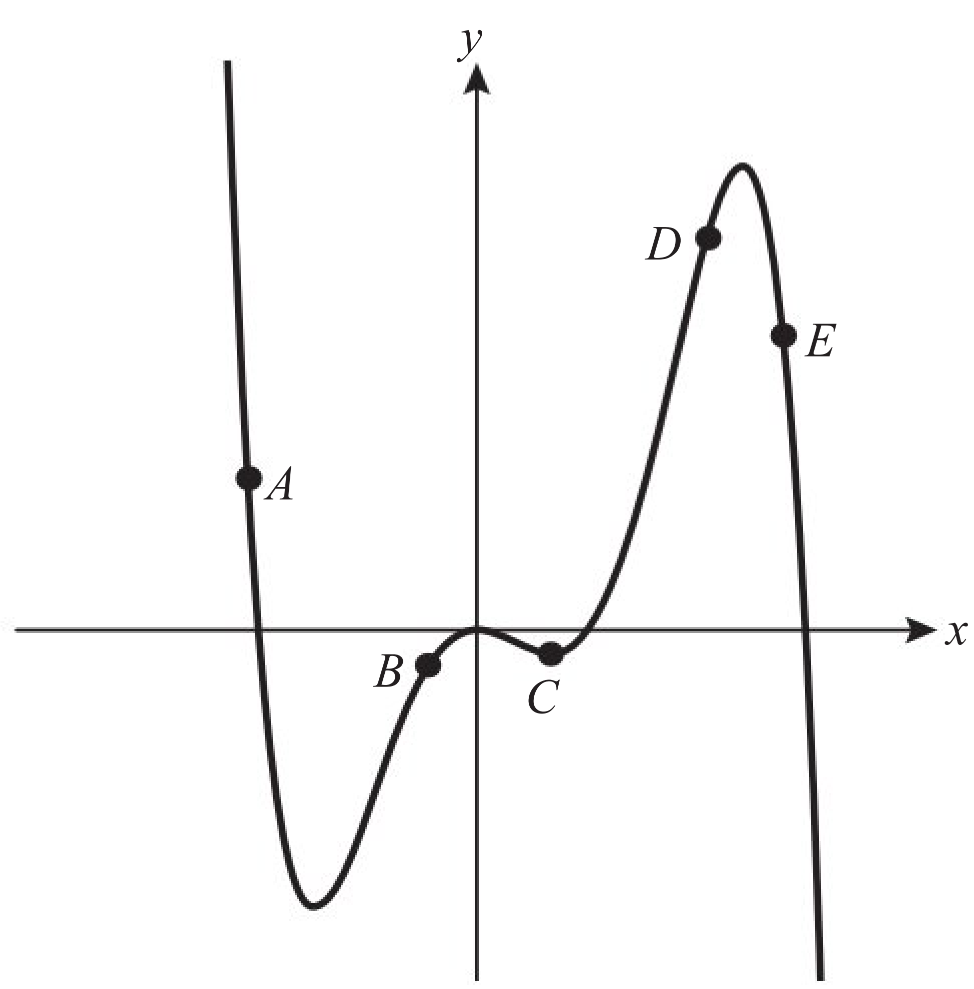
27 Given that $f(x) = \frac{\ln x}{4x}$, show that $f'(x) = \frac{1 - \ln x}{4x^2}$.

28 Given that $y = x^3 - 3 \ln x$, find $\frac{d^2y}{dx^2}$.

29 The diagram shows the graph of $y = f(x)$. Sketch the graph of $y = f''(x)$.



30 For the following graph, write down the points at which the function is concave down.



31 Find the range of values of x for which the function $f(x) = 5x^3 - 2x^2 + 1$ is concave up.

32 The curve with equation $y = x^3 - 24 \ln x$ has a minimum point.
Find its x-coordinate.

33 Show that the graph of $f(x) = \sin x - \cos x$ has a local maximum point at $\left(\frac{3\pi}{4}, \sqrt{2}\right)$.

34 An open box has a square base of side x cm and height $\frac{32}{x^2}$ cm.

Show that the surface area of the box is given by $S = x^2 + \frac{128}{x}$, and find the minimum possible surface area of the box. Show that the value you have found is a minimum.

35 Find the x-coordinate of the point of inflection on the curve with equation $y = 3x^5 - 10x^4 + 8x + 2$.

36 The displacement, s m, of an object at time t seconds, is given by $s = 3\sin(5t)$.
Find the acceleration of the object after 2 seconds.

37 The displacement of an object is given by $s = 3e^{-0.2t}$, where s is measured in metres and t in seconds. Find the speed of the object after 4 seconds.

38 The velocity of an object, measured in $m\,s^{-1}$, is given by $v = \frac{1}{\sqrt{t+3}}$. When $t = 2$, the displacement of the object from the origin is 4 m.
Find the displacement from the origin when $t = 5$.

39 The velocity of an object at time t seconds is given by $v = 2\cos(0.4t)\,m\,s^{-1}$.
Find the distance travelled by the object in the first 10 seconds.

40 Find $\int 2x^{-\frac{2}{3}} + \frac{4}{3x}\,dx$.

41 Find $\int \left(2e^{4x} + 3e^{-\frac{1}{3}x}\right) dx$.

42 Find the following integrals:

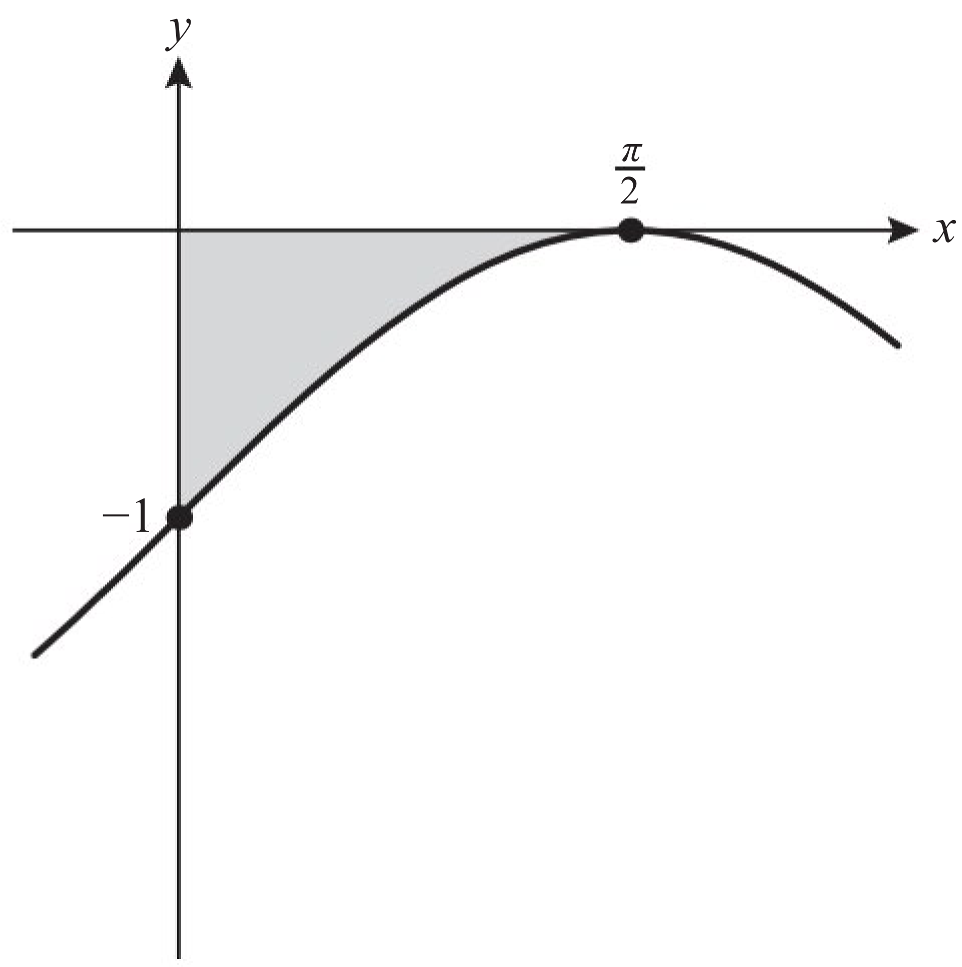
a $\int 4\cos x\ \sin^2 x\,dx$

b $\int \frac{x}{x^2+3}\,dx$

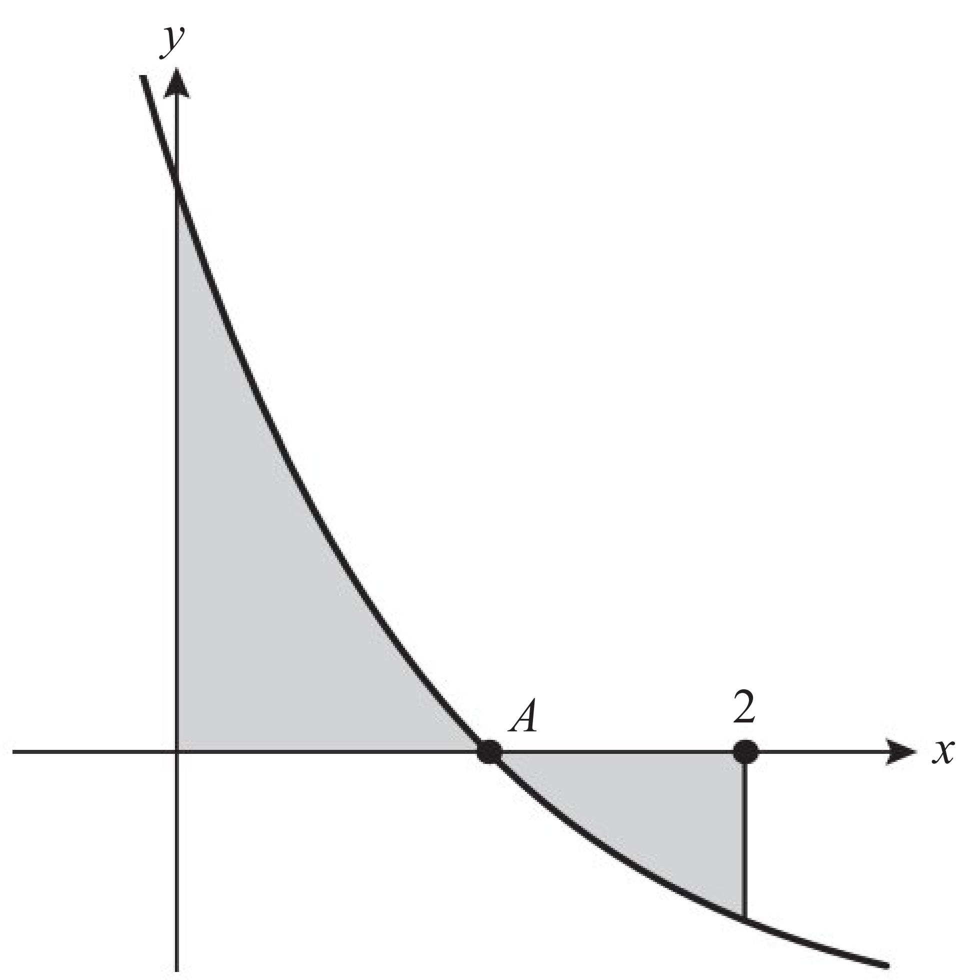
43 Evaluate $\int_0^{\frac{\pi}{6}} \sin 2x \, dx$.

44 Evaluate $\int_0^2 2e^{-x^3} \, dx$.

45 The curve in the diagram has equation $y = \sin x - 1$.
Find the exact value of the shaded area.



46 The curve in the diagram has equation $y = 3e^{-x} - 1$.



a Find the exact coordinates of A.

b Find the shaded area.

47 **a** Find the coordinates of the points of intersection of the curves $y = 2e^{0.5x}$ and $y = x + 5$.

b Find the area enclosed by the two curves.

48 A function is defined by

$$f(x) = \begin{cases} x^2 \text{ for } x \leqslant 3 \\ 5x + k \text{ for } x > 3 \end{cases}$$

Find the value of k so that $f(x)$ is continuous at $x = 3$.

49 Determine whether the function

$$g(x) = \begin{cases} 2x^2 \text{ for } x \leqslant 2 \\ 9\left(x - \frac{2}{3}\right)^2 \text{ for } x > 2 \end{cases}$$

is differentiable at $x = 2$.

50 **a** Show that $\frac{\cos x}{1 - \cos x}$ diverges to infinity as x tends to zero.

b Find $\lim\limits_{x \to \infty} \left(\frac{4x^2 + 3}{3x^2 - 1}\right)$.

51 Use differentiation from first principles to find the derivative of $y = 3x^2 - 4$.

52 For $f(x) = \sin(2x)$, find the exact value of $f^{(3)}\left(\frac{\pi}{6}\right)$.

\

53 a Use l'Hôpital's rule to evaluate

$$\lim_{x \to 0} \frac{e^x - 1}{\sin(2x)}.$$

b Show that $\dfrac{\ln(x-2)}{\tan\left(\frac{\pi x}{4}\right)}$ diverges to infinity when $x \to 2$.

54 Use Maclaurin series to evaluate

$$\lim_{x \to 0} \frac{\cos(3x) - 1}{2x^2}.$$

55 Use l'Hôpital's rule to determine whether

$\lim_{x \to \infty} \dfrac{e^{0.2x}}{2x^2}$ is finite.

56 a Given that $x^2 - 3y^3 = 20$, find an expression for $\dfrac{dy}{dx}$ in terms of x and y.

b Find the gradient of the curve with equation $x \sin y + y \cos x = \dfrac{\pi}{2}$ at the point $\left(0, \dfrac{\pi}{2}\right)$.

57 The area of a circle decreases at the rate of $3\,\text{cm}\,\text{s}^{-1}$.
Find the rate at which the radius of the circle is decreasing at the time when it equals 12 cm.

58 a Find the coordinates of the stationary points on the curve with equation $y = x^3 - 3x^2$.

b Find the maximum value of $x^3 - 3x^2$ for $-3 \leqslant x \leqslant 3$.

59 Find $f'(x)$ for the following:

a $f(x) = x \arctan(2x)$

b $f(x) = \log_2(\sec x)$.

60 Find $\int \operatorname{cosec} x\,(\cot x - \operatorname{cosec} x)\,dx$.

61 Find $\int \frac{2}{\sqrt{4x - 4x^2}}\,dx$.

62 **a** Write $\frac{2x + 5}{x^2 - x - 2}$ in the form $\frac{A}{x - 2} - \frac{B}{x + 1}$.

b Hence find the exact value of

$\int_3^9 \frac{2x + 5}{x^2 - x - 2}\,dx$.

63 Use the substitution $x = 2 \sec u$ to find $\int \frac{1}{x\sqrt{x^2 - 4}}\,dx$.

64 Use the substitution $u = x - 3$ to find the exact value of

$\int_3^7 x\sqrt{x - 3}\,dx$.

65 Find

a $\int x e^{3x}\,dx$.

b $\int x^4 \ln 5x\,dx$.

66 Find $\int x^2 \cos 2x\,dx$.

67 Find the exact value of the area enclosed by the curve with equation $y = 2\ln x$, the y-axis and the lines $y = 2$ and $y = 6$.

68 The region enclosed by the curve $y = \sin x$ and the x-axis, between $x = 0$ and $x = \pi$, is rotated around the x-axis. Find the volume of the resulting solid of revolution.

69 The part of the curve $y = x^2$ between $x = 1$ and $x = 3$ is rotated around the y-axis.
Find the volume of the resulting solid.

70 The speed of an object decreases at a rate proportional to its current velocity.
Write down a differential equation to model this situation.

71 Variables x and y satisfy the differential equation $\frac{dy}{dx} = \sin(x + y)$. When $x = 0$, $y = 2$.
Use Euler's method with step size 0.1 to approximate the value of y when $x = 0.4$. Give your answer to three decimal places.

72 Find the general solution of the differential equation
$\frac{dy}{dx} = xy^2 + x$.

73 Solve the differential equation $\frac{dy}{dx} = (x - 1)(y + 2)$ given that $y = 1$ when $x = 1$.
Express y in terms of x.

74 a Show that the substitution $y = vx$ transforms the differential equation $\frac{dy}{dx} = \frac{y}{x} + \sqrt{\frac{y}{x}}$ into the equation $x\frac{dv}{dx} = \sqrt{v}$.

b Hence solve the equation $\frac{dy}{dx} = \frac{y}{x} + \sqrt{\frac{y}{x}}$ given that $y = 0$ when $x = 1$.
Write y in terms of x.

75 Find the general solution of the differential equation $\frac{dy}{dx} + \frac{2xy}{x^2 + 1} = 4x$, giving your answer in the form $y = f(x)$.

76 Derive the first four non-zero terms of the Maclaurin series for $\ln(1 + x)$.

77 Find the first three non-zero terms in the Maclaurin expansion of $\cos(3x)$.

78 Find the Maclaurin expansion of $\frac{e^x}{1 + x}$, up to and including the x^2 term.

79 Use the series for $\frac{1}{\sqrt{1 - x^2}}$ to obtain the first three non-zero terms of the Maclaurin series for $\arcsin x$.

80 Variables x and y satisfy the differential equation $\frac{dy}{dx} = x + 2y$, and $y(0) = A$.

Write $y = \sum_{k=0}^{\infty} a_k x^k$.

a Express a_1 and a_2 in terms of A and show that, for $k \geqslant 2$, $a_{k+1} = \frac{2}{k+1} a_k$.

b Hence find the Maclaurin series expansion of y.

81 The differential equation $\frac{d^2y}{dx^2} = -\sin y$, with $y = \frac{\pi}{2}, \frac{dy}{dx} = 0$ at $x = 0$.

a Find the values of $\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$ and $\frac{d^4y}{dx^4}$ at $x = 0$.

b Hence find the Maclaurin series for y, up to and including the term in x^4.

Paper plan

	Syllabus Section	Description	Book Section	Mastery: Covered	Mastery: Revised	Mastery: Confident	Practice Paper Coverage: Book P1	Book P2	Book P3	A1	A2	A3	B1	B2	B3	C1	C2	C3
Number and algebra Core	S1.1	Standard form	1B											2				
	S1.2	Arithmetic sequences and series	2A					3		6		1		1	1		10a	
	S1.3	Geometric sequences and series	2B								1b						10b	
	S1.4	Financial applications of geometric sequences	2C					1									10	
	S1.5a	Exponents with integer coefficients	1A															
	S1.5b	Introduction to logarithms	1C							6, 9								
Number and algebra SL	S1.6	Deductive proof	11A									1		4				1
	S1.7a	Rational exponents	12A									1						
	S1.7b	More logarithms	12B					2,7b					1, 6	5		5,8	7	
	S1.8	Infinite geometric sequences	13A								1b					7		
	S1.9	Binomial theorem	13B											10	1			
Number and algebra HL	H1.10a	Counting principles	1A,B					8	2		9						6	1
	H1.10b	Extension of binomial theorem	2A								5							
	H1.11	Partial fractions	2B				6a							5		5	7	
	H1.12	Cartesian form	4A				3			3						2		2
	H1.13	Modulus–argument and Euler form	4B										8					2
	H1.14a	Complex conjugate roots	4C											9				
	H1.14b	Powers and roots of complex numbers	4D										8					2
	H1.14c	Trigonometric identities	4E					12a		12b				4				
	H1.15a	Proof by induction	5A				8		2	12a		1	7		1			1
	H1.15b	Proof by contradiction	5B							9a								
	H1.15c	Disproof by counterexample	5C							9b		1					8	
	H1.16	Systems of linear equations	2C				10b						11a					
Functions Core	S2.1	Equations of straight lines	4A				1						3					
	S2.2	Concepts of functions	3A				5						12cf					
	S2.3	The graph of a function	3B						1			2						
	S2.4	Key features of graphs	3B				12a					2		12d			10bde	

	Syllabus Section	Description	Book Section	Mastery: Covered	Mastery: Revised	Mastery: Confident	Practice Paper Coverage: Book P1	Book P2	Book P3	A1	A2	A3	B1	B2	B3	C1	C2	C3
Functions SL	S2.5a	Composite functions	14A				5			5								
	S2.5b	Finding inverse functions	14B							5						1b		
	S2.6	Quadratic functions	15A							10a						1		
	S2.7a	Quadratic equations and inequalities	15B				7b		1			1				8		
	S2.7b	Quadratic discriminant	15C							10b						4		
	S2.8	Rational functions	16B															
	S2.9	Exponential and logarithmic functions	16C						1				6					
	S2.10a	Solving equations analytically	17A							6, 12d						8		
	S2.10b	Solving equations graphically including using technology	17B								6b			12c			2, 10c, 11c	
	S2.11	Transformations of graphs	16A							10c								
Functions HL	H2.12a	Graphs and equations of polynomials	6A						1	4b								
	H2.12b	The factor and remainder theorems	6B							4a							8	
	H2.12c	Sum and product of roots	6C				7c							9		12c		
	H2.13	More rational functions	7A							8			12ce					
	H2.14	Properties of functions	7E													1a, 13a		
	H2.15	Inequalities	7B					7a		4b						4		
	H2.16a	The modulus function	7C				4a						4, 12ab			3		
	H2.16b	More transformations of graphs	7D				4b			10d						9		
Geometry and trigonometry Core	S3.1a	Distances and midpoints	4B															
	S3.1b	Volume and surface area of 3D solids	5A								1a							
	S3.1c	Angle between intersecting lines and planes	5B											2				
	S3.2a	Trigonometry in right-angled triangles	5B											2				
	S3.2b	Trigonometry in non-right-angled triangles	5B								3			12abcd				
	S3.3	Applications of trigonometry	5C															
Geometry and trigonometry SL	S3.4	Radian measure and application to circles	18A														2	
	S3.5	Extending definitions of trig ratios	18B										3	4				
	S3.6	Trigonometric identities	18C							11a	11b			12d				
	S3.7	Graphs of trigonometric functions	18D						1	12e		2		12c				
	S3.8	Solving trigonometric equations	18E							12c						12b		

	Syllabus Section	Description	Book Section	Mastery			Practice Paper Coverage											
				Covered	Revised	Confident	Book P1	Book P2	Book P3	A1	A2	A3	B1	B2	B3	C1	C2	C3
Geometry and trigonometry HL	H3.9	Reciprocal and inverse trigonometric functions	3A								3			4		10	4	
	H3.10	Compound angle identities	3B				7a,12a	9			3	2		4		12a		
	H3.11	Symmetries of trigonometric graphs	3B									2				10		
	H3.12a	Introduction to vectors	8A					6a										
	H3.12b	Geometry and vectors	8B					6d								11b		
	H3.13	The scalar product	8C								8, 12e							
	H3.14	Equation of a line in 3D	8D					6b			12a		11b			11a		
	H3.15	Intersection of lines	8E					6c			12a							
	H3.16	The vector product	8F								8, 12b					11c		
	H3.17	Equation of a plane	8G								12c		11bc			11c		
	H3.18	Angles and intersections between lines and planes	8H				10				12d		11bc					
Statistics and probability Core	S4.1	Sampling	6A														1a	
	S4.2	Statistical diagrams	6C				2						2					
	S4.3	Summary statistics	6B								10a			3				
	S4.4	Correlation and regression	6D								10b						1bc	
	S4.5	Definitions in probability	7A															1
	S4.6	Probability techniques	7B				2a			1			2, 5		2			
	S4.7	Discrete random variables	8A				10ab	10ce							2		3	
	S4.8	Binomial distribution	8B				2b				10d			11c				
	S4.9	Normal distribution	8C								10d			11ab				
Statistics and probability SL	S4.10	x-on-y regression	19A								10c							
	S4.11	Formal conditional probability	19B										5	11d				
	S4.12	Standardizing normal variables	19C					10a						11a				
Statistics and probability HL	H4.13	Bayes' theorem	9A					10d		1			5	11e				
	H4.14a	Variance of a discrete random variable	9B					10b							2		3b	
	H4.14b	Continuous random variables	9C				10cde				2					13bcde		
	H4.14c	Linear transformation of a random variable									10e				2		3c	

	Syllabus Section	Description	Book Section	Mastery			Practice Paper Coverage											
				Covered	Revised	Confident	Book P1	Book P2	Book P3	A1	A2	A3	B1	B2	B3	C1	C2	C3
Calculus Core	S5.1a	Concept of a limit	9A									1						
	S5.1b	Interpretation of derivatives	9B								4			7				
	S5.2	Increasing and decreasing functions	9B													4		
	S5.3	Derivative of polynomials	9C						1						2	4		
	S5.4	Tangents and normals	9D				1		1				3	6				
	S5.5a	Integration as anti-differentiation	10A															
	S5.5b	Definite integrals and areas using technology	10B								2			8				
Calculus SL	S5.6a	Derivatives of standard functions	20A						1	11a		2	3					
	S5.6b	Chain rule	20B							7				13a	2		9, 12a	
	S5.6c	Product and quotient rules	20C							7			3, 10ab, 12d	13a	2	13d	9	
	S5.7	Second derivative	20D									2	10c	13a				
	S5.8a	Stationary points and optimization	20E				9, 12a		1	11a			10b, 12d	7				
	S5.8b	Points of inflection	20F							11b	4		10c					
	S5.9	Kinematics	21C								6						11	
	S5.10	Integration of standard functions	21A							11c			1	5		13b		2
	S5.11	Further definite integrals and areas	21B							11c			10d		1	13bce	10gh	

	Syllabus Section	Description	Book Section	Mastery			Practice Paper Coverage											
				Covered	Revised	Confident	Book P1	Book P2	Book P3	A1	A2	A3	B1	B2	B3	C1	C2	C3
Calculus HL	H5.12	Limits, continuity and differentiability	10A					5			7				1			
	H5.13	L'Hôpital's rule	10B					5		7								
	H5.14a	Implicit differentiation	10C				9							7				
	H5.15b	Related rates of change	10D														9	
	H5.15c	Optimization	10E												2		5	
	H5.15	Calculus of more complicated functions	10F				1, 6b, 12ab				11ab			5, 13c		5, 13e	7, 12a	
	H5.16a	Integration by substitution	10G							2							7	
	H5.16b	Integration by parts	10H							11c			10d					
	H5.17	Area from y-axis and volume of revolution	10I					4						8				
	H5.18a	First order differential equations and Euler's method	11A					11c			11c							
	H5.18b	Separating variables and homogeneous differential equations	11B								11a			13bc				2
	H5.18c	The integrating factor	11C					11a			11b						12b	
	H5.19a	Maclaurin series	11D				11a	12b					9			6		
	H5.19b	Using Maclaurin series to solve differential equations	11E					11b									12c	

Practice exam papers

Mathematics: analysis and approaches
Higher level
Paper 1 Practice Set A

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in an answer booklet.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A copy of the mathematics: analysis and approaches formula book is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1 *[Maximum mark: 5]*

On his way to school, Suresh stops for coffee with probability 0.8. If he stops for coffee, the probability that he is late for school is 0.4; otherwise, the probability that he is late is 0.1. Given that on a particular day Suresh is late for school, what is the probability that he did not stop for coffee?

2 *[Maximum mark: 7]*

Use the substitution $u = x - 3$ to find the exact value of $\int_3^7 5x\sqrt{x-3}\,dx$.

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3 *[Maximum mark: 6]*

z is the complex number which satisfies the equation $3z - 4z^* = 18 + 21\text{i}$. Find $\left|\frac{z}{3}\right|$.

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4 *[Maximum mark: 6]*

a Show that $(2x + 1)$ is a factor of $f(x) = 2x^3 - 13x^2 + 17x + 12$. *[2]*

b Solve the inequality $2x^3 - 13x^2 + 17x + 12 > 0$. *[4]*

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5 *[Maximum mark: 6]*

Given the functions

$f(x) = \dfrac{2 - x}{x + 3}$ $(x \neq -3)$ and $g(x) = \dfrac{2}{x - 1}$ $(x \neq 1)$

find $(f \circ g)^{-1}$ in the form $\dfrac{ax + b}{cx + d}$.

6 *[Maximum mark: 6]*

Find the possible values of x such that $45e^x$, $7e^{2x}$ and e^{3x} are consecutive terms of an arithmetic sequence.

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7 *[Maximum mark: 6]*

Use L'Hôpital's rule to find

$$\lim_{x \to \pi} \frac{x \sin x}{\ln\left(\frac{x}{\pi}\right)}.$$

8 *[Maximum mark: 7]*

Sketch the graph of

$$y = \frac{2x^2 + 5x - 12}{x + 3}$$

State the coordinates of all axis intercepts and the equations of all asymptotes.

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9 *[Maximum mark: 6]*

a Prove that $\log_2 5$ is an irrational number. *[4]*

b Aron says that $\log_2 n$ is an irrational number for every integer $n \geqslant 10$. Give a counterexample to disprove this statement. *[2]*

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Do **not** write solutions on this page

Section B

Answer **all** questions in an answer booklet. Please start each question on a new page.

10 *[Maximum mark: 20]*

a Sketch the graph of $y = x^2 + 3x - 10$, showing clearly the axes intercepts and the coordinates of the vertex. *[4]*

b **i** Show that the line $y = 2x - 20$ does not intersect the graph of $y = x^2 + 3x - 10$.

ii Find the set of values of k for which the line $y = 2x - k$ intersects the graph of $y = x^2 + 3x - 10$ at two distinct points. *[7]*

c Describe fully a sequence of transformations which transforms the graph of $y = x^2 + 3x - 10$ to the graph of $y = \left(2x + \frac{3}{2}\right)^2 + 2$. *[4]*

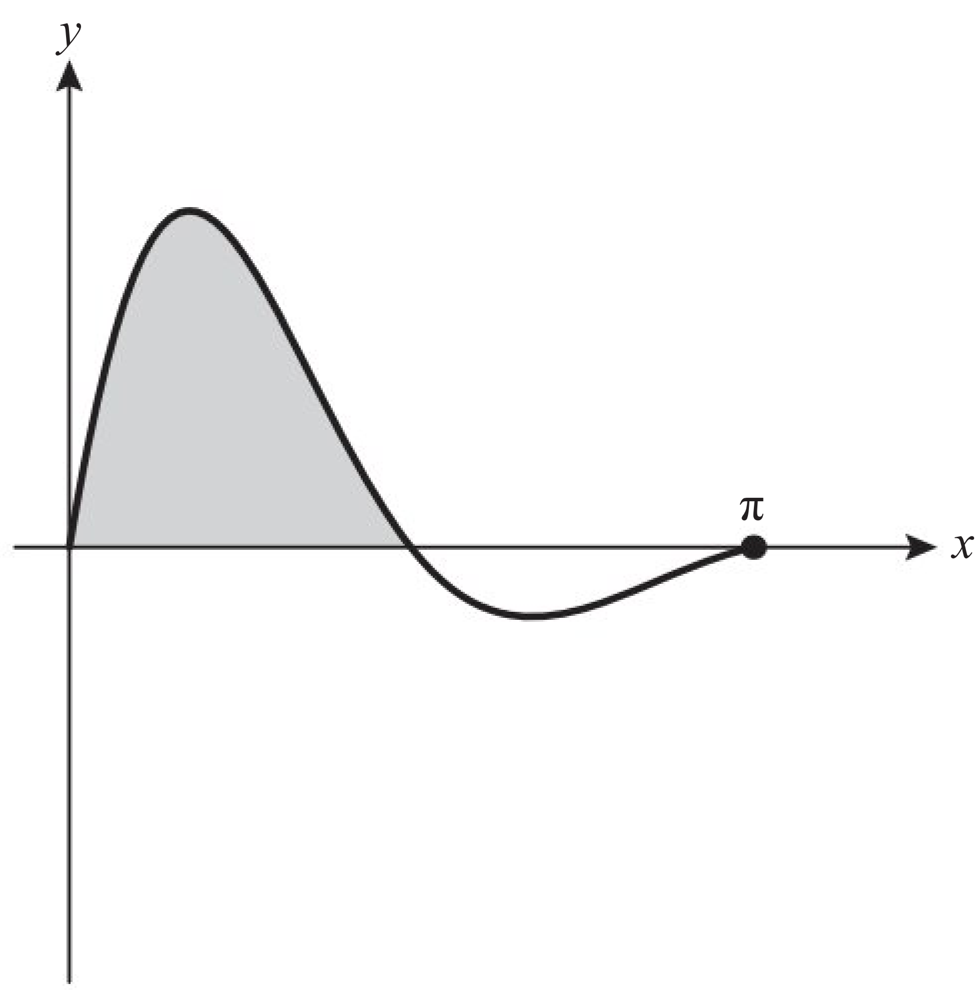
d Sketch the following graphs, indicating clearly all axes intercepts, asymptotes and turning points:

i $y = |x^2 + 3x - 10|$

ii $y = \dfrac{1}{x^2 + 3x - 10}$ *[5]*

11 *[Maximum mark: 16]*

The graph of $y = e^{-x} \sin 2x$ for $0 \leqslant x \leqslant \pi$ is shown below.



The graph has a maximum point at P, a minimum point at Q and points of inflection at R and S.

a Show that the x-coordinates of point P and point Q satisfy
$\tan 2x = 2$. *[4]*

b Show that the x-coordinates of points R and S satisfy
$\tan 2x = -\frac{4}{3}$. *[4]*

c Show that the area of the shaded region enclosed below the curve and above the x-axis is given by $a + be^c$, where a, b and c are constants to be found. *[8]*

12 *[Maximum mark: 19]*

a State and prove de Moivre's theorem. *[5]*

b Use de Moivre's theorem to prove that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$. *[4]*

c Solve the equation $\cos 5\theta = 0$ for $0 \leqslant \theta \leqslant \pi$. *[2]*

d By considering the equation $16c^5 - 20c^3 + 5c = 0$, where $c = \cos\theta$, find the exact value of $\cos\left(\frac{\pi}{10}\right)$.

Justify your choice. *[6]*

e Find the exact value of $\cos\left(\frac{\pi}{10}\right)\cos\left(\frac{7\pi}{10}\right)$. *[2]*

Mathematics: analysis and approaches
Higher level
Paper 2 Practice Set A

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in an answer booklet.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A copy of the mathematics: analysis and approaches formula book is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1 *[Maximum mark: 6]*

A child makes a caterpillar out of modelling clay. The density of the clay is 1.45 g cm^{-3}. She starts by making a sphere of radius 3 cm.

a Find the mass of this sphere. *[2]*

She then adds more spheres, each with radius half the previous one.

b Show that the mass of the caterpillar can never reach 200 g. *[4]*

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2 *[Maximum mark: 6]*

Find the standard deviation of a continuous random variable with the probability density function

$$f(x) = \begin{cases} 0.4106 \sin x \sqrt{x - 2\pi} & \text{for } 2\pi \leqslant x \leqslant 3\pi \\ 0 & \text{otherwise} \end{cases}.$$

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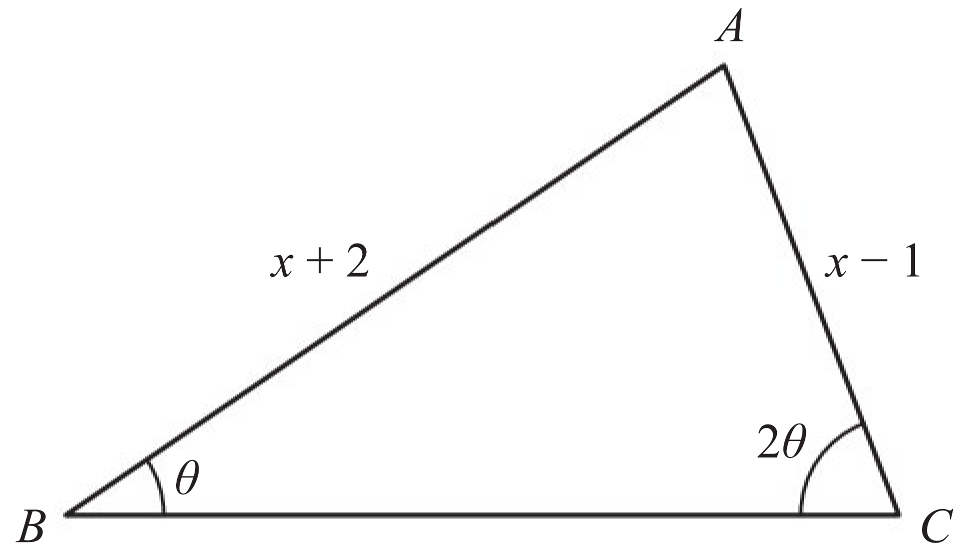
3 *[Maximum mark: 7]*

A triangle has sides of length $(x - 1)$ and $(x + 2)$, and angles θ and 2θ, as shown in the diagram.



Show that $x = \dfrac{4}{4 - 3\sec^2\left(\frac{\theta}{2}\right)}$.

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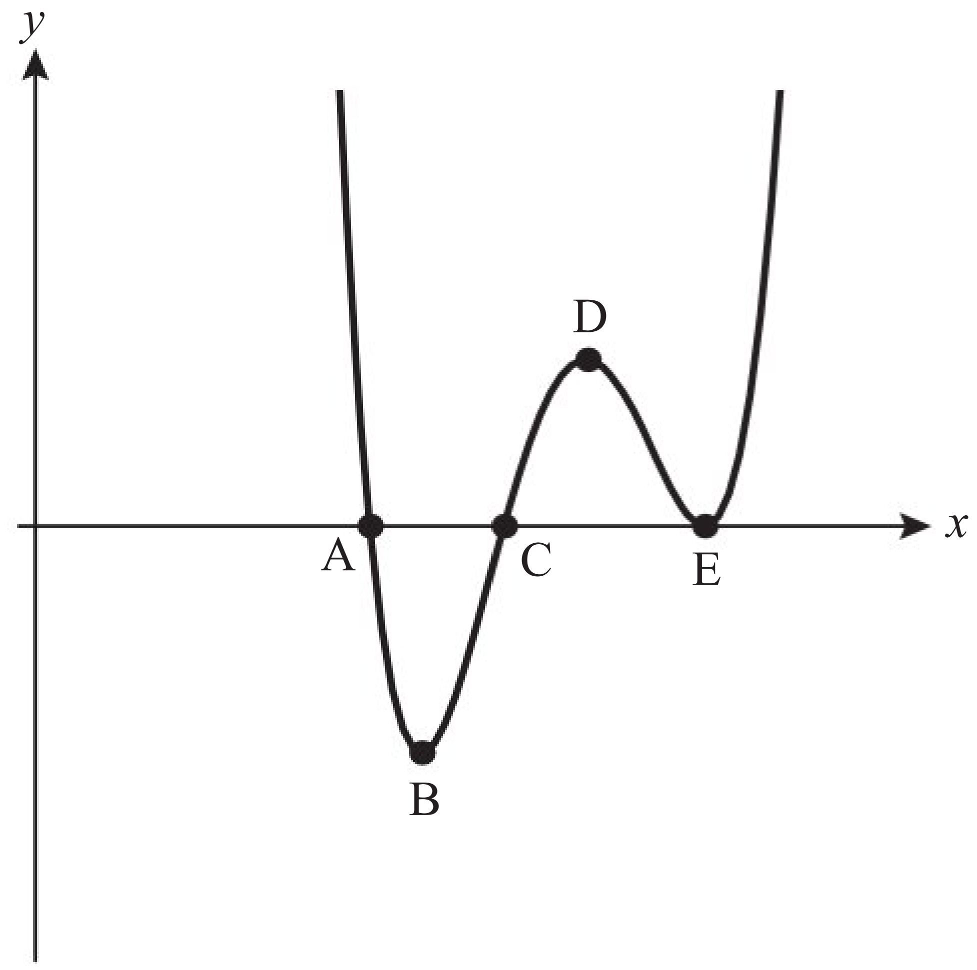
4 *[Maximum mark: 5]*

The graph of $y = f'(x)$ is shown in the diagram.



Write down the labels of the following points, justifying your choice in each case:

a local maximum point(s) of $f(x)$ *[2]*

b point(s) of inflection of $f(x)$. *[3]*

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5 *[Maximum mark: 6]*

Find the first four terms in the binomial expansion of $\frac{1}{\sqrt{4-x}}$, and state the set of values of x for which the expansion is valid.

6 *[Maximum mark: 6]*

A particle moves in a straight line, with the velocity at time t seconds given by $v = \frac{\sin t}{\sqrt{t+1}}\,\mathrm{m\,s^{-1}}$.

a Find the distance travelled by the particle in the first five seconds of motion, giving your answer to one decimal place. *[3]*

b Find the first two times when the magnitude of acceleration is $0.3\,\mathrm{m\,s^{-2}}$. *[3]*

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7 *[Maximum mark: 6]*

Find the values of b and c such that the function

$$f(x) = \begin{cases} x^2 - 1 \text{ for } x \leqslant 1 \\ -x^2 + bx + c \text{ for } x > 1 \end{cases}$$

is both continuous and differentiable.

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8 *[Maximum mark: 5]*

Vectors **a** and **b** satisfy

$\mathbf{a} \cdot \mathbf{b} = 17$ and $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$.

Find, in degrees, the size of the angle between the two vectors.

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9 *[Maximum mark: 7]*

Ten children, including two pairs of (non-identical) twins, line up in a straight line for a photo.

a Find the number of arrangements in which each pair of twins stands together. *[3]*

b The photographer arranges the children at random. Find the probability that neither pair of twins stands together. *[4]*

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Do **not** write solutions on this page

Section B

Answer **all** questions in an answer booklet. Please start each question on a new page.

10 *[Maximum mark: 21]*

The marks of Miss Rahman's class of twelve students on Mathematics Paper 1 and Paper 2 are given in the table.

Student	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*
Paper 1	72	105	98	106	63	58	52	87	75	72	91	68
Paper 2	72	87	91	98	68	56	61	72	73	61	97	52

a Find the mean and standard deviation of each set of marks. Hence write two comments comparing the marks on the two papers. *[4]*

b The critical value of the Pearson's correlation coefficient for 12 pieces of data is 0.532. Determine whether there is significant positive correlation between the two sets of marks. *[3]*

c Two students did not sit Paper 1.

i Student 13 scored 86 marks on Paper 2. Use an appropriate regression line to estimate what mark he would have got on Paper 1.

ii Student 14 scored 45 marks on Paper 2. Can your regression line be used to estimate her mark for Paper 1? Justify your answer. *[5]*

d It is known that, in the population of all the students in the world who took Paper 1, the marks followed the distribution $N(68, 11^2)$. It is also known that 12% of all students achieved Grade 7 in this paper.

i How many of the 12 students in Miss Rahman's class achieved Grade 7 in Paper 1?

ii Find the probability that, in a randomly selected group of 12 students, there are more Grade 7s than in Miss Rahman's class. *[6]*

e Paper 1 is marked out of 110. In order to compare the results to another paper, Miss Rahman rescales the marks so that the maximum mark is 80. Find the mean and standard deviation of the rescaled Paper 1 marks for the 12 students in the class. *[3]*

11 *[Maximum mark: 16]*

a Find the general solution of the differential equation $\frac{dy}{dx} - y\tan x = 0$, expressing y in terms of x. *[5]*

b Consider the differential equation $\frac{dy}{dx} - y\tan x = \cos x$.

i Show that the integrating factor is $\cos x$.

ii Hence find the general solution of the differential equation. *[7]*

c Consider instead the differential equation $\frac{dy}{dx} - y^2\tan x = \cos x$ with the initial condition $y = 2$ when $x = 0$.

Use Euler's method with step length 0.1 to estimate the value of y when $x = 0.5$. *[4]*

12 *[Maximum mark: 19]*

The lines l_1 and l_2 have equations

$$l_1 : \mathbf{r} = \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ 2 \\ -3 \end{pmatrix}$$

$$l_2 : \mathbf{r} = \begin{pmatrix} 1 \\ -8 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$$

a **i** Show that the lines l_1 and l_2 intersect.

ii Find the coordinates of the point of intersection, P. *[7]*

b Find a vector perpendicular to l_1 and l_2. *[2]*

c Hence find the equation of the plane Π containing l_1 and l_2. Give your answer in the form $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$. *[2]*

The line l_3 passes through the point $Q(-11, 0, 1)$ and intersects Π at the point P.

d Find the exact value of the sine of the acute angle between l_3 and Π. *[6]*

e Hence find the shortest distance from the point Q to Π. *[2]*

Mathematics: analysis and approaches
Higher level
Paper 3 Practice Set A

Candidate session number

1 hour

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A copy of the mathematics: analysis and approaches formula book is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1 *[Maximum mark: 30]*

This question is about investigating and proving properties of a sequence called the Fibonacci sequence.

The Fibonacci sequence is defined by the initial conditions $F_1 = 1, F_2 = 1$

and the recursion relation

$F_{n+2} = F_n + F_{n+1}$ for $n \geqslant 1$.

a Write down F_3, F_4 and F_5. *[3]*

b Dominique suggests that 1 is the only Fibonacci number which is a perfect square.
Use a counterexample to disprove this statement. *[2]*

c Prove by induction that

$$\sum_{i=1}^{i=n} (F_i)^2 = F_n F_{n+1}.$$ *[6]*

d Find the smallest value of k such that $F_k \geqslant k$. Prove that $F_n \geqslant n$ for $n \geqslant k$. *[7]*

e It is suggested that $F_n = \alpha^n$ might satisfy the recursion relation. Given that $\alpha \neq 0$, find the two possible values of α. *[4]*

f Show that if α_1 and α_2 are the two possible values of α then $F_n = A\alpha_1^n + B\alpha_2^n$, where A and B are constants, also satisfies the recursion relation. *[2]*

g Find an expression for F_n in terms of n. *[4]*

h Hence find the value of $\frac{F_{n+1}}{F_n}$ as n tends to infinity. *[2]*

2 *[Maximum mark: 25]*

This question is about resonance in vibrating objects.

a Write down the period of the function $\cos \pi t$. *[1]*

b **i** Sketch the function $y = \cos \pi t + \cos 2\pi t$ for $0 \leqslant t \leqslant 3$.

ii Write down the period of the function $\cos \pi t + \cos 2\pi t$. *[2]*

c **i** Use technology to investigate the period of the given functions below. Write down the values of A, B and C.

$\mathbf{f}(t)$	**Period**
$\cos \pi t + \cos 1.5\pi t$	A
$\cos \pi t + \cos 1.25\pi t$	B
$\cos \pi t + \cos 1.1\pi t$	C

ii Hence conjecture an expression for the period, T, of $f(t) = \cos \pi t + \cos\left(\left(1 + \frac{1}{n}\right)\pi t\right)$ where n is an integer. *[4]*

d Prove that, for your conjectured value of T, $f(t + T) = f(t)$. *[3]*

e **i** Use the compound angle formula to write down and simplify an expression for $\cos(A + B) + \cos(A - B)$.

ii Hence find a factorized form for the expression $\cos P + \cos Q$. *[3]*

f By considering the factorized form of $f(t)$, explain the shape of its graph. *[3]*

g A piano string oscillates when plucked. The displacement, x, from equilibrium as a function of time is modelled by:

$$\frac{d^2x}{dt^2} + 4x = 0.$$

Show that a function of the form $x = f(t) = \cos(\omega t)$ solves this differential equation for a positive value of ω to be stated. *[4]*

h The piano string can be subjected to an external driving force from a tuning fork. The differential equation becomes:

$$\frac{d^2x}{dt^2} + 4x = \cos kt.$$

Find a solution of the form $x = f(t) + g(k) \cos kt$ where $g(k)$ is a function to be found. *[3]*

i Resonance is a phenomenon in which the amplitude of the driven oscillation grows without limit. For what positive value of k will resonance occur? Justify your answer. *[2]*

Mathematics: analysis and approaches
Higher level
Paper 1 Practice Set B

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in an answer booklet.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A copy of the mathematics: analysis and approaches formula book is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1 *[Maximum mark: 7]*

Find the value of $a > 0$ such that $\int_0^a \frac{4x}{x^2+3}\,dx = \ln 16$.

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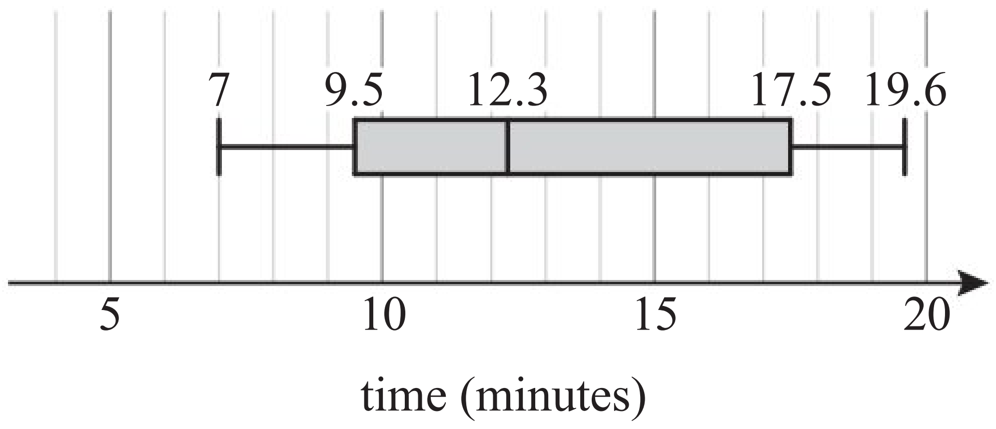
2 *[Maximum mark: 6]*

The box plot summarizes the times taken by a group of 40 children to complete an obstacle course.



Two of the 40 children are selected at random.

a Find the probability that both children completed the course in less than 9.5 minutes. *[3]*

b Find the probability that one child completed the course in less than 9.5 minutes and the other in between 9.5 and 17.5 minutes. *[3]*

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3 *[Maximum mark: 5]*

Find the equation of the normal to the graph of $y = \frac{\sin x}{x}$ at the point where $x = \pi$.

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4 *[Maximum mark: 5]*

Solve the inequality $|x - 3| \leqslant |2x + 1|$.

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5 *[Maximum mark: 6]*

Given that $P(A) = 0.3$, $P(B|A) = 0.6$ and $P(A \cup B) = 0.8$, find $P(A|B)$. Give your answer as a simplified fraction.

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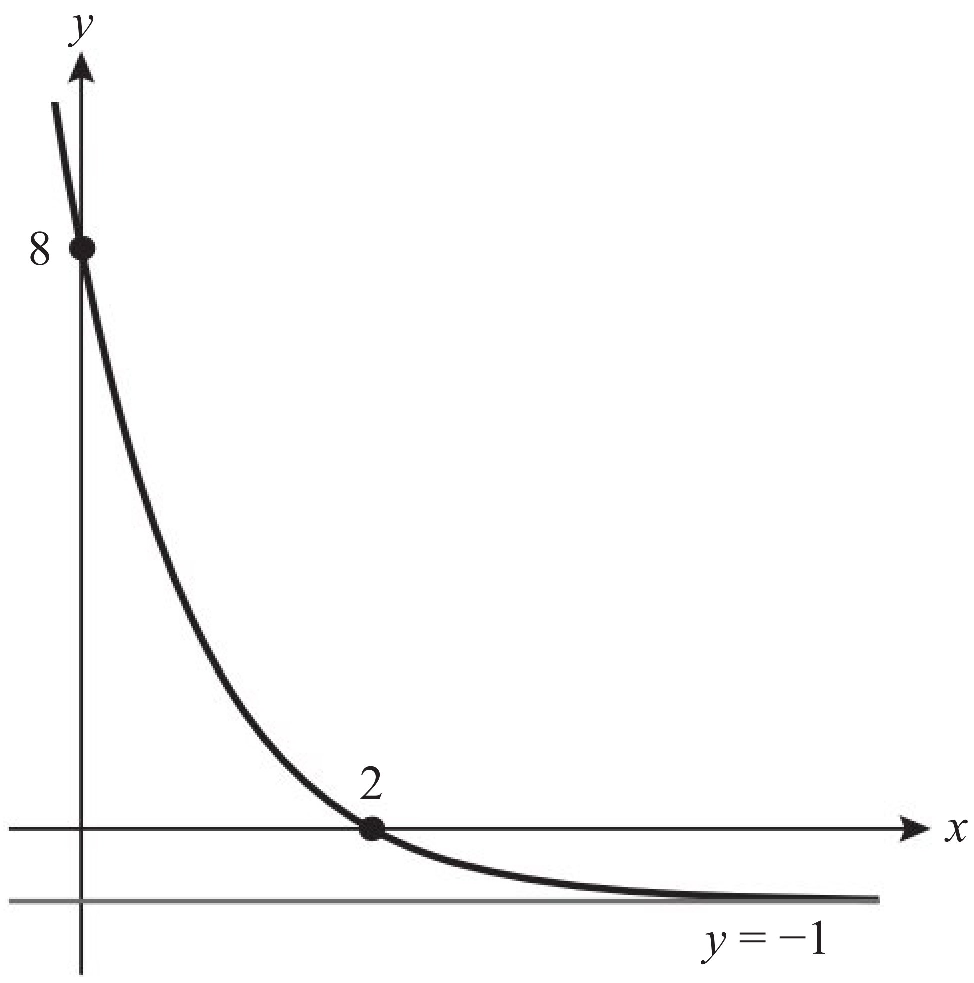
6 *[Maximum mark: 6]*

The graph in the diagram has equation $y = A + Be^{-kx}$.



Find the values of A, B and k.

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7 *[Maximum mark: 6]*

Use mathematical induction to prove that $7^n + 3^{n-1}$ is divisible by 4 for all integers $n \geqslant 1$.

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8 *[Maximum mark: 7]*

Find, in the form $z = re^{i\theta}$, the roots of the equation $z^3 = 4 - (4\sqrt{3})i$.

9 *[Maximum mark: 7]*

Find the first two non-zero terms in the Maclaurin series for $\frac{\cos x}{\sqrt{1-x^2}}$.

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Do **not** write solutions on this page

Section B

Answer **all** questions in an answer booklet. Please start each question on a new page.

10 *[Maximum mark: 18]*

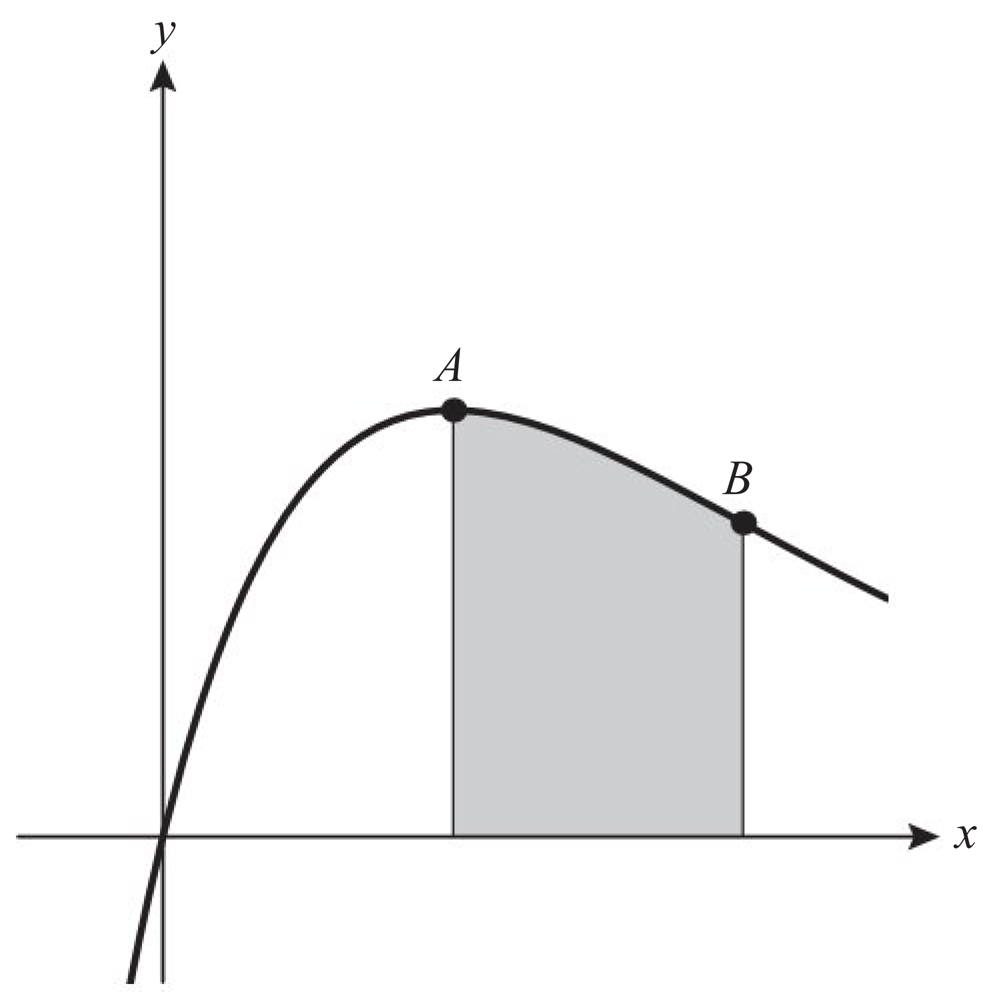
Let $f(x) = xe^{-kx}$ where $x \in \mathbb{R}$ and $k > 0$.

a Show that $f'(x) = (1 - kx)e^{-kx}$ and find $f''(x)$ in the form $(a + bx)e^{-kx}$. *[5]*

b Find the x-coordinate of the stationary point of $f(x)$ and show that it is a maximum. *[5]*

c Find the coordinates of the point of inflection of $f(x)$. *[3]*

d The graph of $y = f(x)$ is shown below. A is the maximum point and B is the point of inflection. Show that the shaded area equals $\frac{2e - 3}{k^2e^2}$. *[5]*



11 *[Maximum mark: 15]*

The following system of equations does not have a unique solution.

$$\begin{cases} 6x + ky + 2z = a \\ 6x - y - z = 7 \\ 2x - 3y + z = 1 \end{cases}$$

a Find the value of k. *[6]*

Each equation represents a plane.

b Find

i the value of a for which the three planes intersect in a line

ii the equation of the line. *[7]*

c If the value of a is such that the three planes do not intersect in a line, describe their geometric configuration, justifying your answer. *[2]*

12 *[Maximum mark: 22]*

Let $f(x) = x^2 - 2x - 3$, $x \in \mathbb{R}$.

a Sketch the graph of $y = |f(x)|$. *[3]*

b Hence or otherwise, solve the inequality $|f(x)| > -\frac{1}{2}x + 4$. *[6]*

Let $g(x) = \frac{2x - 7}{f(x)}$.

c State the largest possible domain of g. *[1]*

d Find the coordinates of the turning points of g. *[5]*

e Sketch the graph of $y = g(x)$, labelling all axis intercepts and asymptotes. *[5]*

f Hence find the range of g for the domain found in part **c**. *[2]*

Mathematics: analysis and approaches
Higher level
Paper 2 Practice Set B

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in an answer booklet.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A copy of the mathematics: analysis and approaches formula book is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

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Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1 *[Maximum mark: 5]*

In an arithmetic sequence, the fifth term is 7 and the tenth term is 81. Find the sum of the first 20 terms.

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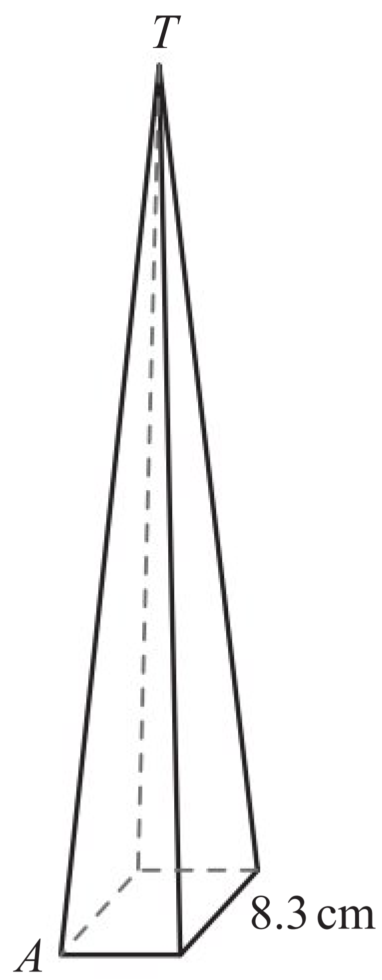
2 *[Maximum mark: 4]*

A flag pole has the shape of a square-based pyramid shown in the diagram. The side length of the base is 8.3 cm. The edge AT makes an angle of 89.8° with the base.



Find the height of the flag pole. Give your answer in centimetres, in standard form, correct to two significant figures.

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3 *[Maximum mark: 5]*

In this question, an *outlier* is defined as a piece of data which is more than two standard deviations above or below the mean.

The heights of eight children, in centimetres, are:

122	124	127	131	134	134	136	147

Determine whether any of the heights are outliers.

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4 *[Maximum mark: 5]*

Prove that

$$\frac{\sin\left(x+\frac{\pi}{3}\right)-\sin\left(x-\frac{\pi}{3}\right)}{\cos\left(x+\frac{\pi}{3}\right)-\cos\left(x-\frac{\pi}{3}\right)} \equiv -\cot x$$

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5 *[Maximum mark: 6]*

Find

$$\int \frac{2}{x(x-1)}\,dx$$

Write your answer in the form $\ln(f(x)) + c$.

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6 *[Maximum mark: 6]*

The normal to the graph of $y = 4\sin^{-1}\left(\frac{x}{2}\right)$ at the point where $x = 1.5$ intersects the x-axis at the point A and the y-axis at the point B. Find the area of triangle AOB, where O is the origin.

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7 *[Maximum mark: 7]*

Find the coordinates of the points on the curve

$$3x^2 + 2xy - y^2 + 24 = 0$$

where the tangent is parallel to the x-axis.

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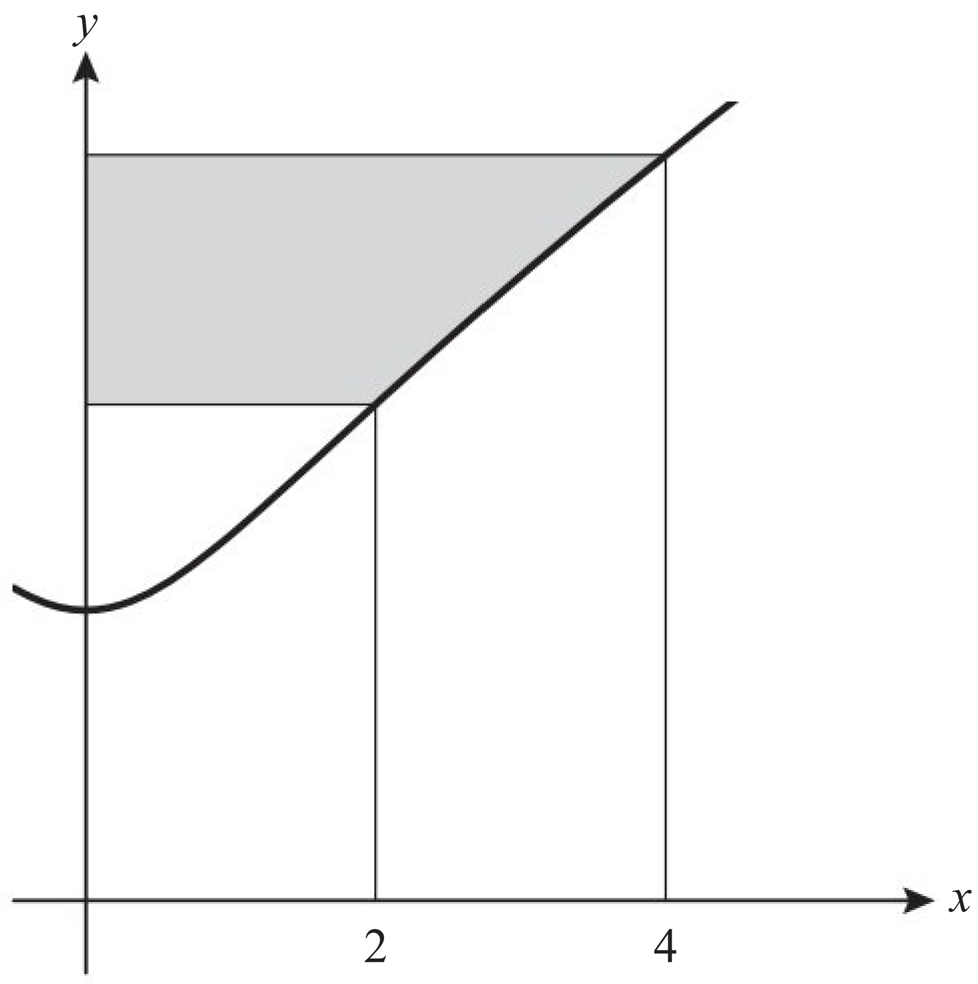
8 *[Maximum mark: 7]*

The curve in the diagram has equation $y = \sqrt[3]{x^2 + 1}$. The shaded region is bounded by the curve, the y-axis and two horizontal lines.



a Find the area of the shaded region. *[4]*

b Find the volume generated when the shaded region is rotated 2π radians about the y-axis. *[3]*

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9 *[Maximum mark: 5]*

One of the roots of the equation $x^3 - 7x^2 + bx + c = 0$ is $2 - \text{i}$. Find the value of c.

10 *[Maximum mark: 5]*

The constant term in the binomial expansion of $\left(x^2 + \frac{1}{x}\right)^n$ is 495. Find the value of n.

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Do **not** write solutions on this page

Section B

Answer **all** questions in an answer booklet. Please start each question on a new page.

11 *[Maximum mark: 18]*

Battery life of a certain brand of smartphone can be modelled by a normal distribution with mean μ hours and standard deviation σ hours. It is known that 5% of the batteries last less than 24 hours, while 20% last more than 72 hours.

a **i** Show that $\mu + 0.8416\sigma = 72$ and find another similar equation connecting μ and σ.

ii Show that approximately 65.7% of the batteries last longer than 48 hours. *[7]*

b Find the interquartile range of battery life, giving your answer to the nearest hour. *[2]*

c Find the probability that, out of 20 randomly selected batteries, at least 10 last more than 48 hours. *[3]*

d Given that a battery has lasted for 48 hours, what is the probability that it will last for another 24 hours? *[2]*

e A customer buys a new smartphone and tests the battery. If the battery lasts less than 24 hours they return the phone with probability 0.9. If it lasts between 24 and 72 hours they return the phone with probability 0.2. Otherwise they do not return the phone.

Given that a customer keeps the phone, what is the probability that its battery lasts more than 72 hours? *[4]*

12 *[Maximum mark: 18]*

Two of the sides of a triangle have length x cm and $2x$ cm, and the angle between them is $\theta°$. The perimeter of the triangle is 10 cm.

a In the case $x = 2$, find the area of the triangle. *[4]*

b Explain why x must be less than $\frac{10}{3}$. *[2]*

c **i** Show that $\cos\theta = \frac{15x - x^2 - 25}{x^2}$.

ii Sketch the graph of $y = \frac{15x - x^2 - 25}{x^2}$ for $x > 0$.

iii Hence find the range of possible values of x. *[7]*

d Find the value of x for which the triangle has the largest possible area, and state the value of that area. *[5]*

13 *[Maximum mark: 19]*

Consider the differential equation $\frac{dy}{dx} = \frac{y}{x+y}$.

a Find and simplify an expression for $\frac{d^2y}{dx^2}$ in terms of x and y. *[7]*

b Show that the substitution $y = xv$ transforms this equation into $x\frac{dv}{dx} = f(v)$, where $f(v)$ is a function to be found. *[4]*

c Hence find the particular solution of the equation $\frac{dy}{dx} = \frac{y}{x+y}$ for which $y = 1$ when $x = 1$. Give your answer in the form $x = g(y)$. *[8]*

Mathematics: analysis and approaches
Higher level
Paper 3 Practice Set B

Candidate session number

1 hour

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A copy of the mathematics: analysis and approaches formula book is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1 *[Maximum mark: 30]*

This question is about using sums of sequences to investigate the formula for integrating a polynomial.

a Prove by induction that

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}.$$ *[7]*

b Simplify $(n+1)^3 - n^3$. *[2]*

c By considering two different ways of expressing

$$\sum_{r+1}^{n} (r+1)^3 - r^3$$

show that

$$\sum_{1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}.$$ *[7]*

d By considering splitting the region into n rectangles each of width $\frac{x}{n}$ and whose top right corner lies on the curve $y = x^2$, show that

$$\int_0^x t^2 \, dt \leqslant \frac{x}{n} \sum_{r=1}^{n} \left(\frac{rx}{n}\right)^2.$$ *[4]*

e By considering rectangles whose top left corner lies on the curve $y = x^2$, form a similar inequality to provide a lower bound on $\int_0^x t^2 \, dt$. *[4]*

f By considering the limit as $n \to \infty$ prove that

$$\int_0^x t^2 \, dt = \frac{x^3}{3}.$$ *[6]*

2 *[Maximum mark: 25]*

This question is about estimating parameters from data.

Let X_1 and X_2 both be random variables representing independent observations from a population with mean μ and variance σ^2.

You may use without proof in this question the fact that

$$E(aX_1 + bX_2) = a\,E(X_1) + b\,E(X_2)$$

and

$$\mathrm{Var}(aX_1 + bX_2) = a\,\mathrm{Var}(X_1) + b\,\mathrm{Var}(X_2).$$

a Find an expression for $\bar{X}$, the random variable representing the sample mean of the two observed values. *[1]*

b Show that $E(\bar{X}) = \mu$ and find an expression for $\mathrm{Var}(\bar{X})$ in terms of σ. *[4]*

The sample variance is defined as

$$S^2 = \frac{X_1^2 + X_2^2}{2} - \bar{X}^2$$

c **i** Find $E(X^2)$ in terms of $\mathrm{Var}(X)$ and $E(X)$.

ii Show that $E(S^2) = \frac{1}{2}\sigma^2$. *[4]*

An unbiased estimator of a population parameter is one whose expected value equals the population parameter.

d **i** Show that $M = \frac{2X_1 + 3X_2}{5}$ is an unbiased estimator of μ.

ii When comparing two unbiased estimators, the one with a lower variance is said to be more efficient. Determine whether M or $\bar{X}$ is a more efficient unbiased estimator of μ. *[5]*

In a promotion, tokens are placed at random in boxes of cereal. Y is the random variable describing the number of boxes of cereal that need to be opened, up to and including the one where a token is found. Two independent investigations were conducted.

e The tokens are placed in cereal boxes with probability p. The presence of a token in a cereal box is independent of other boxes.

i Find an expression for L, the probability of observing $Y_1 = a$ and $Y_2 = b$ in terms of a, b and p.

ii Find the value of p which maximizes L. This is called the maximum likelihood estimator of p. *[8]*

In the first observation, Y was found to be 4. In the second observation Y was found to be 8.

f **i** Find an unbiased estimate for the variance of Y.

ii Find a maximum likelihood estimate for p. *[3]*

Mathematics: analysis and approaches
Higher level
Paper 1 Practice Set C

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in an answer booklet.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A copy of the mathematics: analysis and approaches formula book is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1 *[Maximum mark: 6]*

Let $f(x) = 2x^2 + 10x + 7, x \in \mathbb{R}$.

a Find the largest possible domain of the form $x \leqslant k$ for which the inverse function, f^{-1}, exists. *[2]*

b For the value of k from part **a**, find the inverse function $f^{-1}(x)$, stating its domain. *[4]*

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2 *[Maximum mark: 6]*

Let $z = 3 - 2\text{i}$ and $w = -1 + \text{i}$.

a Represent z and w on an Argand diagram. *[2]*

b Find $\frac{w}{z}$ in the form $a + b\text{i}$. *[2]*

c Find the real numbers p and q such that $pz + qw = 6$. *[2]*

3 *[Maximum mark: 5]*

Solve the inequality $|2x + 1| < |x - 3|$.

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4 *[Maximum mark: 5]*

Find the set of values of k for which the function $f(x) = x^3 + kx^2 + kx - 2$ is strictly increasing for all $x \in \mathbb{R}$.

5 *[Maximum mark: 6]*
Evaluate

$$\int_1^6 \frac{3x - 16}{3x^2 + 10x - 8}\,dx$$

Give your answer in the form $\ln k$.

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6 *[Maximum mark: 5]*

a Use Maclaurin series to find constant a such that $\frac{1}{10}\sin 3x \approx ax$ when $x \approx 0$. *[2]*

b Hence find the approximate solutions of the equation $\frac{1}{10}\sin 3x = x^2$. *[3]*

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7 *[Maximum mark: 5]*

The sum of the first two terms of a geometric series is 3 and its sum to infinity is 5.

Given that all terms of the series are positive, find the common ratio of the series.

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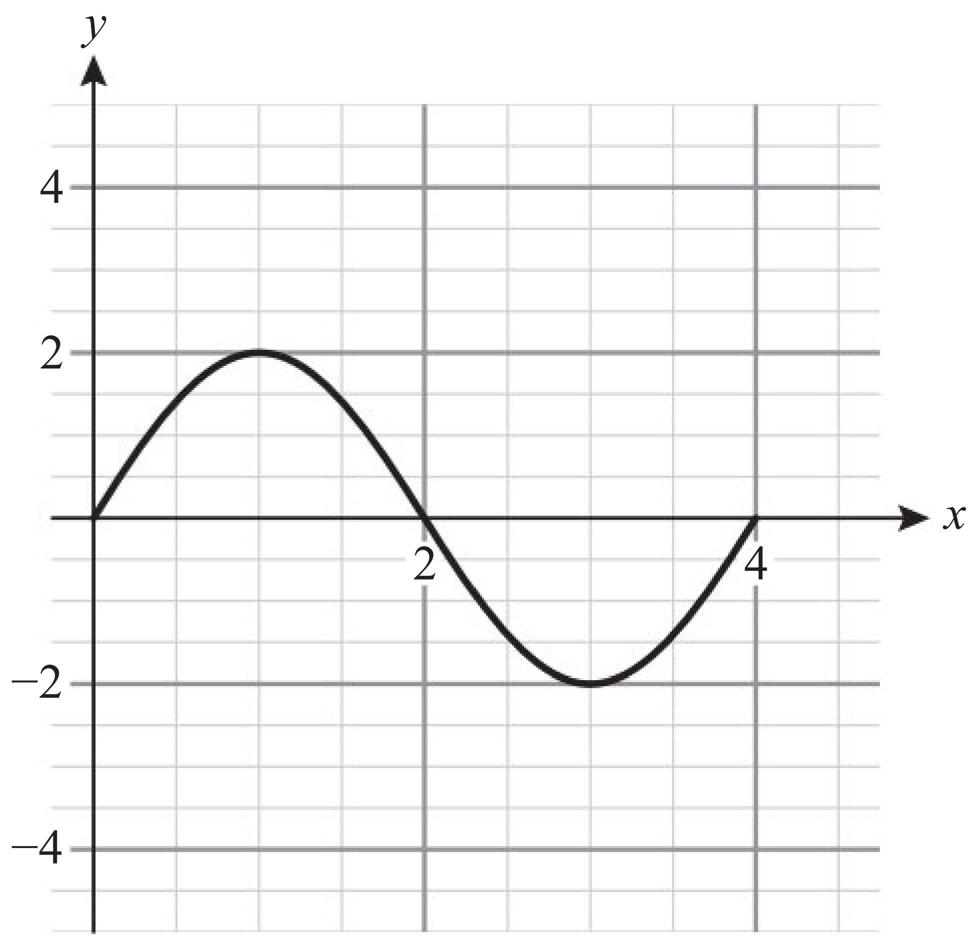
8 *[Maximum mark: 7]*

Solve the equation

$$\log_4(3 - 2x) = \log_{16}(6x^2 - 5x + 12).$$

9 *[Maximum mark: 5]*

The graph of $y = f(x)$ is shown in the diagram. The domain of f is $0 \leqslant x \leqslant 4$.



a On the same grid, sketch the graph of $y = [f(x)]^2$. *[3]*

b Find the domain and range of the function $g(x) = 2f(x - 1)$. *[2]*

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10 *[Maximum mark: 5]*

Let $y = \arcsin x$.

a Express $\arccos x$ in terms of y. *[3]*

b Hence show that $\arcsin x + \arccos x \equiv k$, where k is a constant to be found. *[2]*

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Do **not** write solutions on this page

Section B

Answer **all** questions in an answer booklet. Please start each question on a new page.

11 *[Maximum mark: 18]*

a Points A, B and D have coordinates $A(1, -4, 3)$, $B(2, 1, -1)$ and $D(-1, 3, 3)$.

i Find the equation of the line l_1 through A and B.

ii Write down the equation of the line l_2 which passes through D and is parallel to AB. *[5]*

b **i** Find the exact distance AB.

ii Find the coordinates of two possible points C on the line l_2 such that $CD = 2AB$.

iii Denote the two possible points C by C_1 and C_2. Determine whether angle C_1AC_2 is acute, right or obtuse. *[8]*

c **i** Find $\overrightarrow{AB} \times \overrightarrow{AD}$.

ii Hence find the equation of the plane containing the points A, B and D. *[5]*

12 *[Maximum mark: 16]*

a Use compound angle identities to express $\cos 3\theta$ in terms of $\cos \theta$. *[4]*

b Consider the equation $8x^3 - 6x + 1 = 0$.

i Given that $x = \cos \theta$, for $0 \leqslant \theta \leqslant \pi$, find the value of $\cos 3\theta$.

ii Hence find the possible values of x and show that they are all distinct. *[7]*

c Show that $8\cos\left(\frac{2\pi}{9}\right)\cos\left(\frac{4\pi}{9}\right) = -\sec\left(\frac{8\pi}{9}\right)$. *[3]*

d State, with a reason, the value of $\cos\left(\frac{2\pi}{9}\right) + \cos\left(\frac{4\pi}{9}\right) + \cos\left(\frac{8\pi}{9}\right)$. *[2]*

13 *[Maximum mark: 21]*

Let $f(x) = \dfrac{x}{1 + x^2}$ for $x \in \mathbb{R}$.

a Determine algebraically whether f is an even function, an odd function or neither. *[3]*

The continuous random variable X has probability density function given by

$$g(x) = \begin{cases} \dfrac{kx}{1+x^2} & \text{for } 0 \leqslant x \leqslant \sqrt{3} \\ 0 & \text{otherwise} \end{cases}.$$

b Show that $k = \dfrac{1}{\ln 2}$. *[4]*

c Find the median of X. *[4]*

d Find the mode of X. *[5]*

e Find the mean of X. *[5]*

Mathematics: analysis and approaches
Higher level
Paper 2 Practice Set C

Candidate session number

2 hours

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Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1 *[Maximum mark: 5]*

Sacha is investigating the relationship between time spent doing homework and time spent on social media. In her year at school, at the time of the survey, 60% of students are aged 17 and the rest are aged 18. Sacha wants to represent both age groups fairly, so she takes a random sample of six 17-year-olds and four 18-year-olds.

a State the name of this sampling technique. *[1]*

The results are shown in the table, showing the number of hours per day spent on each activity.

Student	1	2	3	4	5	6	7	8	9	10
Time spent on social media (x)	1.7	3.5	2.6	1.7	2.1	3.2	3.8	2.5	3.1	3.6
Time spent on homework (y)	4.2	2.1	3.2	3.5	4.2	2.5	0.6	2.5	2.7	1.5

Sacha finds that there is a strong negative correlation between the two variables, and decides to use a linear regression line to model the relationship between them.

b Find the equation of the regression line in the form $y = ax + b$. *[2]*

c Interpret, in the context of this question, the meaning of the coefficients a and b. *[2]*

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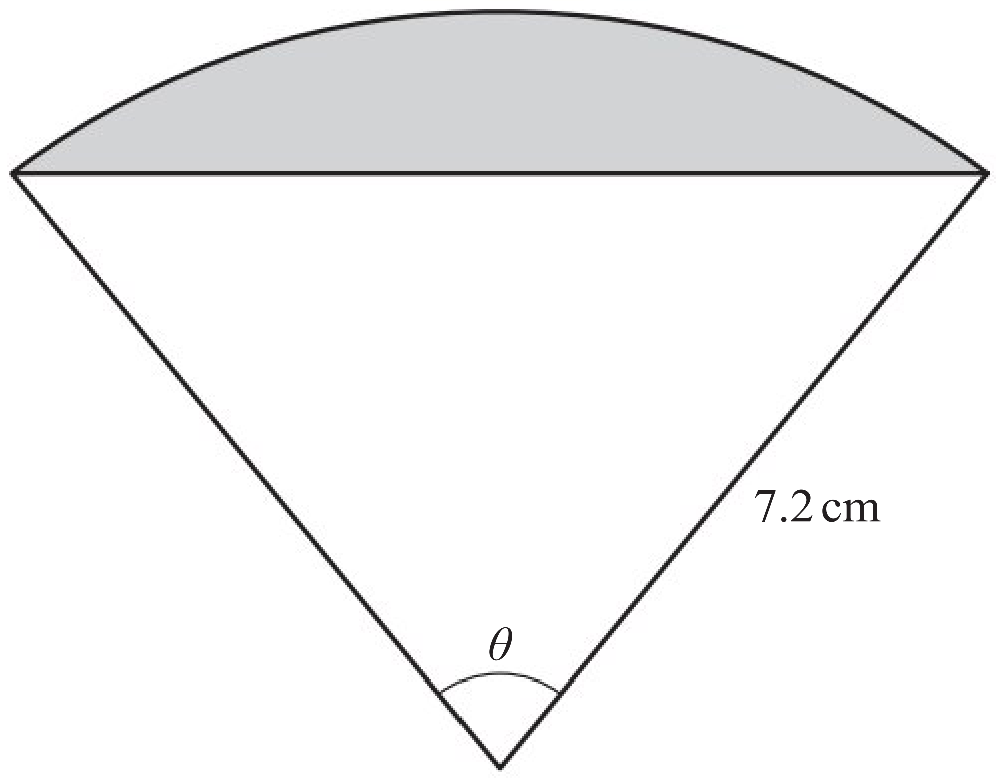
2 *[Maximum mark: 5]*

The diagram shows a sector of a circle with radius 7.2 cm. The angle at the centre is θ radians.



The area of the shaded region is 9.7 cm^2. Find the value of θ.

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3 *[Maximum mark: 8]*

The discrete random variable X has the following probability distribution:

x	1	2	4	7
$\mathbf{P}(X = x)$	k	$2k$	$3k$	$4k$

a Find the value of k. *[2]*

b Find the variance of X. *[4]*

c Find the variance of $20 - 5X$. *[2]*

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4 *[Maximum mark: 5]*

Prove the identity

$$\frac{\sec\theta\ \sin\theta}{\tan\theta + \cot\theta} \equiv \sin^2\theta.$$

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5 *[Maximum mark: 6]*

Ilya decorates small cakes in his bakery. The time it takes him to decorate x cakes is modelled by $T(x) = 0.003x^3 + 10x + 200$ minutes.

a If Ilya wants to work at most 12 hours in a day, what is the largest number of cakes he can decorate? *[2]*

b Ilya can choose to decorate between 20 and 40 cakes per day, inclusive. Find the minimum and maximum time he can take per cake on average. *[4]*

6 *[Maximum mark: 6]*

Ibrahim has 20 different books on his bookshelf. He wants to select six books to take on holiday.

a How many different selections can he make? *[2]*

Out of the 20 books, 12 are fiction and 8 non-fiction. Ibrahim wants to take at least three fiction and at least two non-fiction books.

b How many different selections of six books can he make? *[4]*

7 *[Maximum mark: 7]*

Use the substitution $u = e^x$ to find

$$\int \frac{e^x}{e^{2x} + e^x - 2}\,dx.$$

Give your answer in the form $\ln(f(x))$.

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8 *[Maximum mark: 7]*

Prove by contradiction that there exists no function

$f(x) = 2x^3 + bx^2 + cx + 3$, where $b, c \in \mathbb{Z}$

such that $(2x + 3)$ is a factor of $f(x)$, and when $f(x)$ is divided by $x - 2$ the remainder is 5.

9 *[Maximum mark: 6]*

The total surface area of a cone with radius r and height h is given by $S = \pi r^2 + \pi r\sqrt{r^2 + h^2}$. A cone has height 5 cm and the radius increases at the rate of 2 cm per second. Find the rate at which the surface area is increasing when the radius equals 10 cm.

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Section B

Answer **all** questions in an answer booklet. Please start each question on a new page.

10 *[Maximum mark: 20]*

a Stella is planning to start a small business selling cosmetics gift boxes. She plans to start by selling 30 boxes in the first month. In each subsequent month she plans to sell 10 more boxes than in the previous month.

i According to Stella's plan, how many boxes will she sell in the 12th month?

ii How many boxes will she sell in the first year?

iii In which month will she sell her 2000th box? *[8]*

b Giulio also sells cosmetics gift boxes. He also sells 30 boxes in the first month, but expects to increase his sales by 10% each month.

i How many boxes will Giulio sell in the first year?

ii In which month will Giulio first sell more than 100 boxes? *[6]*

c Stella makes a profit of £2.20 per box and Guilio makes a profit of £3.10 per box.

i Find the profit each person makes in the first year.

ii In which month will Giulio's **total** profit first overtake Stella's? *[6]*

11 *[Maximum mark: 17]*

The velocity (in m s^{-1}) of an object at t seconds is given by

$$v(t) = \frac{8 - 3t}{t^2 - 6t + 10},\ 0 \leqslant t \leqslant 10.$$

Find

a the initial speed *[1]*

b the maximum speed *[2]*

c the length of time for which the speed is greater than $1\,\text{m s}^{-1}$ *[3]*

d the time at which the object changes direction *[2]*

e the length of time for which the object is decelerating *[2]*

f the acceleration after 5 seconds *[2]*

g the distance travelled after 10 seconds *[2]*

h the time when the object returns to its starting position. *[3]*

12 *[Maximum mark: 18]*

a Show that

$$\frac{d}{dx}(\ln|\sec x + \tan x|) = \sec x.$$ *[3]*

b Find the general solution to the differential equation

$$\cos x \frac{dy}{dx} + y = 1,\ -\frac{\pi}{2} < x < \frac{\pi}{2}.$$ *[7]*

Consider now the differential equation

$$\frac{d^2y}{dx^2} + \cos x \frac{dy}{dx} + y = 1,\ -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

c **i** Show that

$$\frac{d^3y}{dx^3} = (\sin x - 1)\frac{dy}{dx} - \cos x \frac{d^2y}{dx^2}.$$

ii Given that $y = 2$ and $\frac{dy}{dx} = 1$ when $x = 0$, find the Maclaurin series solution up to and including the term in x^3. *[8]*

Mathematics: analysis and approaches
Higher level
Paper 3 Practice Set C

Candidate session number

1 hour

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A copy of the mathematics: analysis and approaches formula book is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1 *[Maximum mark: 25]*

This question is about investigating two sequences involving paired parentheses and using links between them to generate a formula.

In computer science, a useful validity check for mathematical expressions is to check that all open parentheses match closed parentheses.

A_n is the number of possible expressions (not necessarily correct) with n pairs of parentheses. For example, when $n = 1$ there are two possible expressions:

() or)(

So $A_1 = 2$.

a Show that $A_2 = 6$. *[2]*

b When $n = 8$:

i How many characters are in the expression?

ii How many of these positions should be chosen to contain open '(' parentheses?

iii Hence find A_8 and state a general expression for A_n. *[4]*

B_n is the number of correct expressions with n pairs of parentheses.

For example, when $n = 1$ the only correct expression is (), so $B_1 = 1$.

c **i** Show that $B_2 = 2$.

ii Find B_3. *[3]*

You are given that $B_8 = 1430$.

d There is a relationship between A_n and B_n of the form

$$B_n = \mathrm{f}(n)\, A_n.$$

Suggest an appropriate function for f(n) and hence suggest an expression for B_n. *[3]*

e It can be shown that $B_{n+1} = \dfrac{4n+2}{n+2} B_n$. Use this result to prove by induction your conjecture from part **d**. *[8]*

f A corrupted computer file includes an expression involving 20 pairs of brackets. The corruption results in the order of the brackets being switched at random.

What is the probability that the end result still has brackets which form a 'correct expression'? *[2]*

g Asher and Elsa are running in an election. 100 voters cast their vote one at a time, and a running total is kept. At the end of the voting the result is a tie with each candidate getting 50 votes.

What is the probability that Asher is never ahead of Elsa at any point during the count? Justify your answer. *[3]*

2 *[Maximum mark: 30]*

This question is about the path of three snails chasing after each other.

a Find $|\mathrm{e}^{\frac{2\mathrm{i}\pi}{3}} - 1|$. *[3]*

Three snails – Alf, Bill and Charlotte – are positioned on the vertices of an equilateral triangle whose centre of rotational symmetry is the origin of the Argand plane. Alf is positioned at the point $z = 1$ and Bill is above the real axis.

b Find the positions of the other two snails. *[2]*

c If Bill is stationary and Alf moves towards him at speed 1 unit per second, how far does Alf travel until he reaches Bill? How long does it take Alf to get there? *[2]*

Alf chases Bill, Bill chases Charlotte and Charlotte chases Alf. They all travel with speed 1 unit per second. The position of Alf at time t is denoted by z_A and the position of Bill is denoted by z_B.

d Explain why $\frac{dz_A}{dt} = \frac{z_B - z_A}{|z_B - z_A|}$. *[2]*

e Write z_B in terms of z_A. *[1]*

f If $z_A = re^{i\theta}$, find an expression for $\frac{dz_A}{dt}$ in terms of r, θ, $\frac{dr}{dt}$ and $\frac{d\theta}{dt}$. *[2]*

g Hence, by comparing real and imaginary parts, find differential equations for $\frac{dr}{dt}$ and $\frac{d\theta}{dt}$. *[7]*

h Solve these differential equations. *[7]*

i How long does it take Alf to reach Bill? How far has Alf travelled until he reaches Bill? How many rotations does he make around the origin? *[4]*

Practice Set A: Paper 1 Mark scheme

SECTION A

1 P(late) = 0.8 × 0.4 + 0.2 × 0.1 (= 0.34) (M1)
P(late and not coffee) = 0.2 × 0.1 (= 0.02) (M1)
P(not coffee|late) M1
$= \frac{0.02}{0.34}$ A1
$= \frac{1}{17}$ A1
[5 marks]

2 Substitute dx = du, $5x = 5(u + 3)$ M1
Change limits M1
Obtain $\int_0^4 5(u+3)\sqrt{u}\,du$ A1
Expand the brackets before integrating: $\int_0^4 5u^{\frac{3}{2}} + 15u^{\frac{1}{2}}\,du$ M1
$= \left[2u^{\frac{5}{2}} + 10u^{\frac{3}{2}}\right]_0^4$ A1
$= 2 \times 2^5 + 10 \times 2^3$ (M1)
$= 144$ A1
[7 marks]

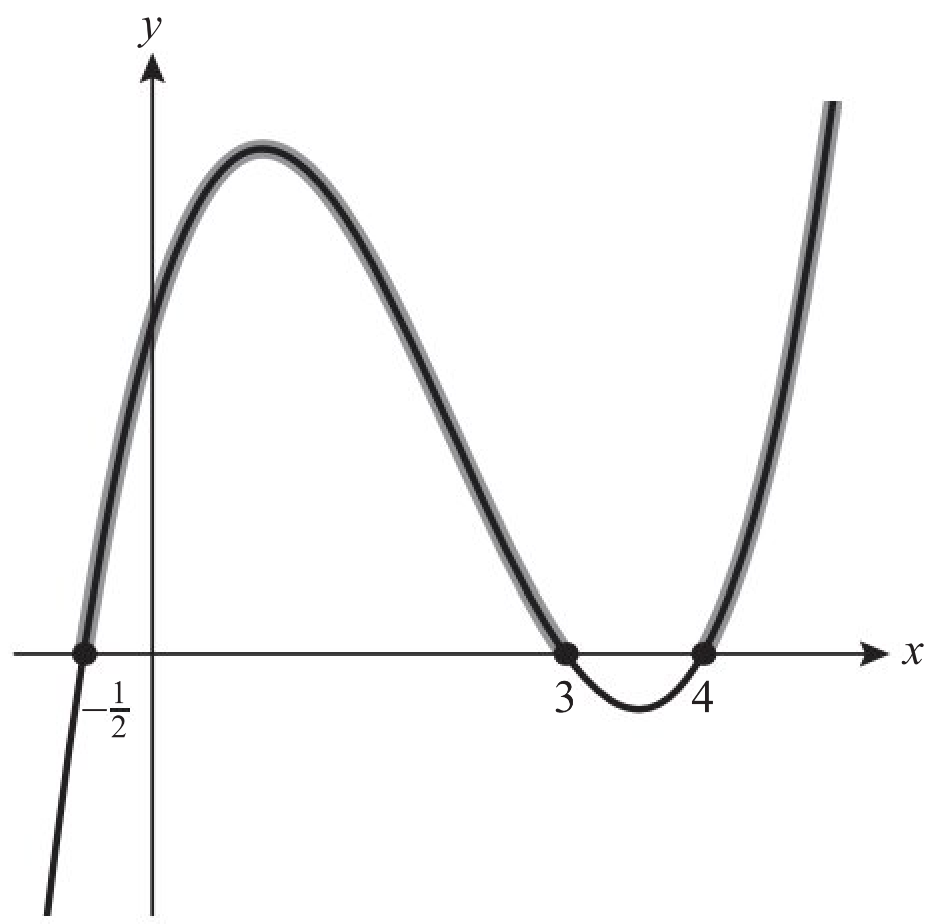
3 Write $z = x + iy$ (M1)
Then $3x + 3iy - 4x + 4iy = 18 + 21i$ A1
Compare real and imaginary parts M1
$z = -18 + 3i$ A1
$\left|\frac{z}{3}\right| = \sqrt{6^2 + 1^2}$ M1
$= \sqrt{37}$ A1
[6 marks]

4 **a** EITHER
Use factor theorem:
$f\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right)^3 - 13\left(-\frac{1}{2}\right)^2 + 17\left(-\frac{1}{2}\right) + 12$ M1
$= -\frac{1}{4} - \frac{13}{4} - \frac{34}{4} + \frac{48}{4}$
$= 0$
So $(2x + 1)$ is a factor A1
OR
Compare coefficients or long division:
$2x^3 - 13x^2 + 17x + 12 = (2x + 1)(x^2 - 7x + 12)$ M1A1

b $(2x + 1)(x - 3)(x - 4) = 0$

$x = -\frac{1}{2}, 3, 4$ (M1)

Sketch graph or consider sign of factors (M1)



$-\frac{1}{2} < x < 3$ or $x > 4$

Note: Award M1A0 for correct region from their roots M1A1

[6 marks]

5 $f \circ g(x) = \dfrac{2 - \dfrac{2}{x-1}}{\dfrac{2}{x-1} + 3}$ M1

$= \dfrac{2(x-1) - 2}{2 + 3(x-1)}$ (M1)

$= \dfrac{2x - 4}{3x - 1}$ A1

$x = \dfrac{2y - 4}{3y - 1}$

$3xy - x = 2y - 4$ (M1)

$3xy - 2y = x - 4$ M1

$y = \dfrac{x - 4}{3x - 2}$ A1

[6 marks]

6 $7e^{2x} - 45e^{x} = e^{3x} - 7e^{2x}$ M1

$e^{3x} - 14e^{2x} + 45e^{3x} = 0$ A1

$e^{x}(e^{x} - 9)(e^{x} - 5) = 0$ M1A1

Reject $e^{x} = 0$ R1

$x = \ln 5$ or $\ln 9$ A1

[6 marks]

7 Attempt to differentiate both top and bottom. M1

Top: $\sin x + x \cos x$ M1A1

Bottom: $\frac{1}{x}$ A1

$\lim_{x \to \pi}(x \sin x + x^2 \cos x)$ M1

$= -\pi^2$ A1

[6 marks]

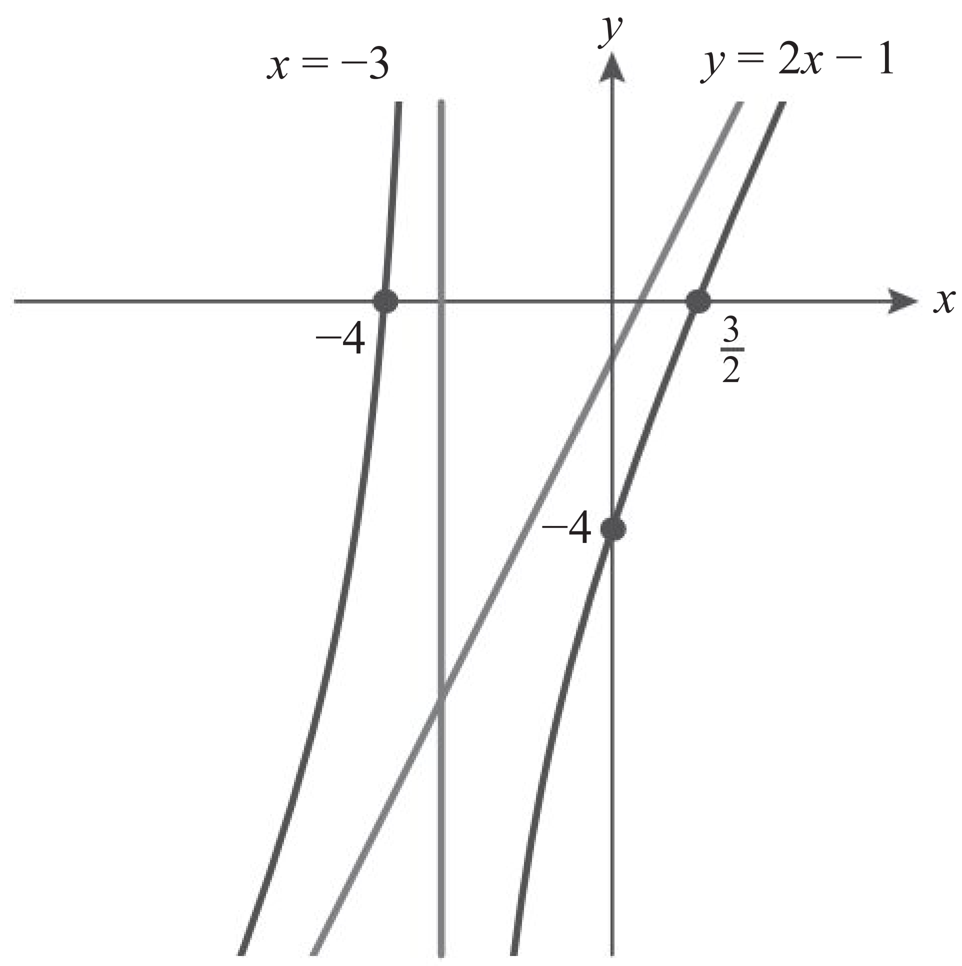
8 Factorize denominator to find x-intercepts: $(2x-3)(x+4)$ (M1)

Long division or compare coefficients:

$$\frac{2x^2+5x-12}{x+3}=\frac{(x+3)(2x-1)-9}{x+3}$$ M1

$$=2x-1-\frac{9}{x+3}$$ A1

Correct shape A1



Axis intercepts: $\left(\frac{3}{2}, 0\right)$, $(-4, 0)$, $(0, -4)$ A1

Vertical asymptote: $x=-3$ A1

Oblique asymptote: $y=2x-1$ A1

[7 marks]

9 **a** Suppose that $\log_2 5$ is rational, and write $\log_2 5=\frac{p}{q}$. M1

Then $2^{\frac{p}{q}}=5$, so $2^p=5^q$. M1

e.g. LHS is even and RHS is odd. A1

This is a contradiction, so $\log_2 5$ is irrational A1

b Any suitable example, e.g. $n=16$ M1

Complete argument, e.g. $\log_2 16=4$, which is rational A1

[6 marks]

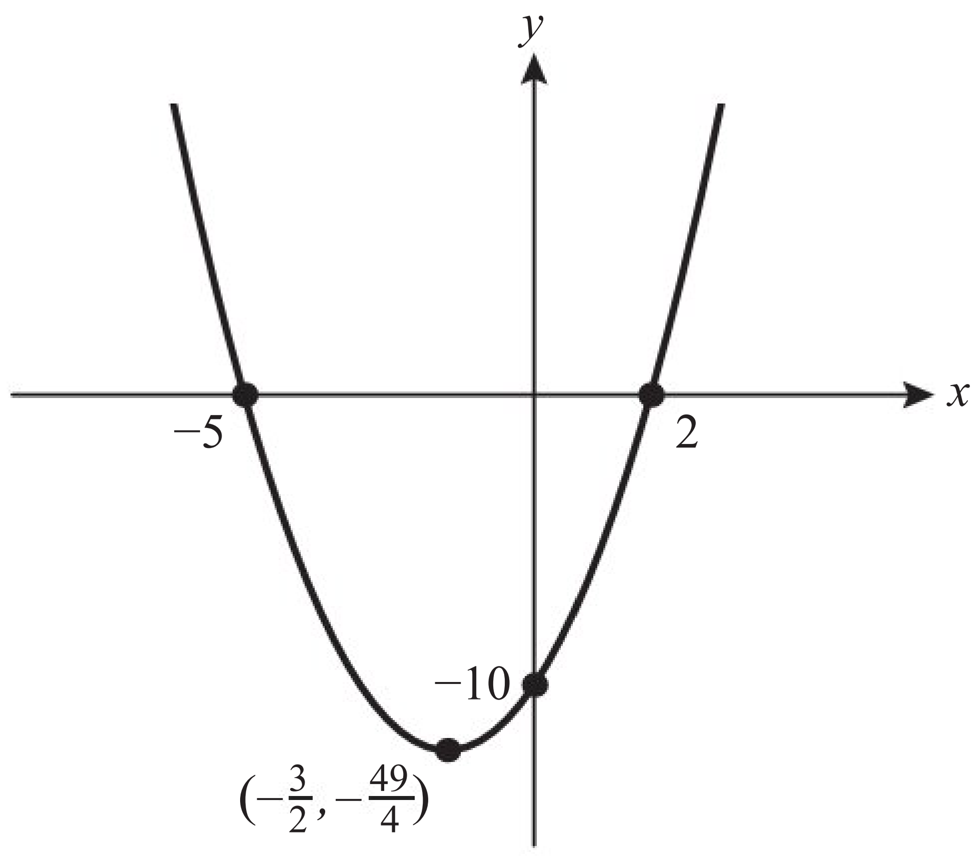
SECTION B

10 a Factorize to find x-intercepts: $(x + 5)(x - 2)$ (M1)

Complete the square for vertex (or half-ways between intercepts):

$\left(x+\frac{3}{2}\right)^2-\frac{49}{4}$ (M1)

Correct shape and all intercepts A1



Correct vertex $\left(-\frac{3}{2}, -\frac{49}{4}\right)$ A1

[4 marks]

b i $x^2 + 3x - 10 = 2x - 20$

$\Leftrightarrow x^2 + x + 10 = 0$ M1

discriminant $= 1 - 40$ $(= -39)$ M1

< 0 so no intersections A1

ii $x^2 + x + (k - 10) = 0$ M1A1

$1 - 4(k - 10) > 0$ M1

$k < \frac{41}{4}$ A1

[7 marks]

c Compare to $\left(x+\frac{3}{2}\right)^2-\frac{49}{4}$ (M1)

Vertical translation $\frac{57}{4}$ units up A1

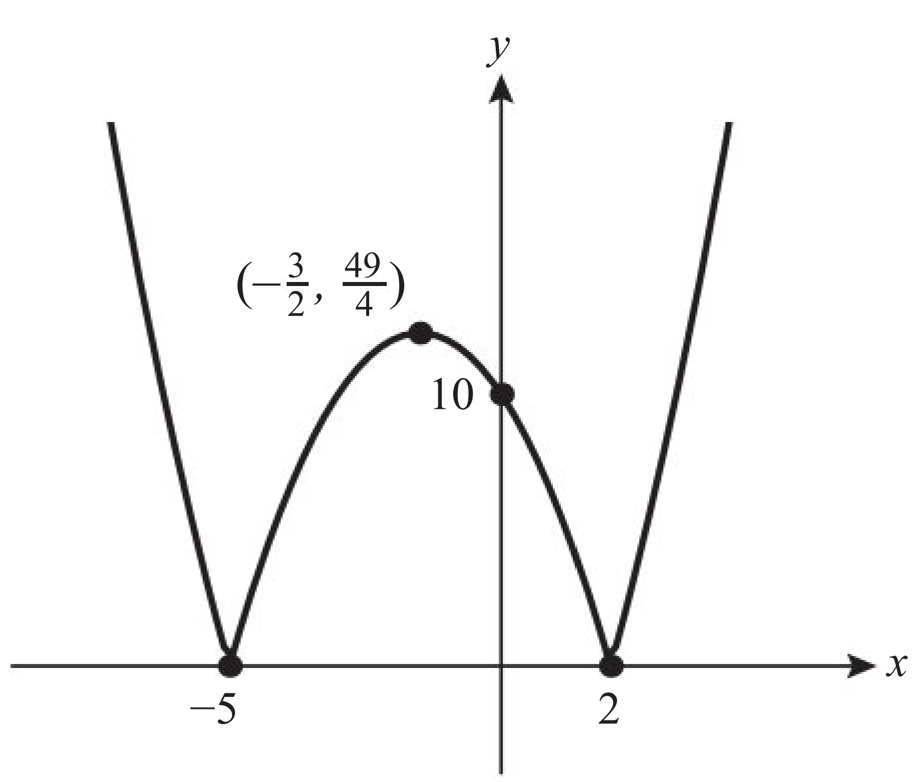
Horizontal stretch A1

Scale factor $\frac{1}{2}$ A1

[4 marks]

d i Correct shape A1

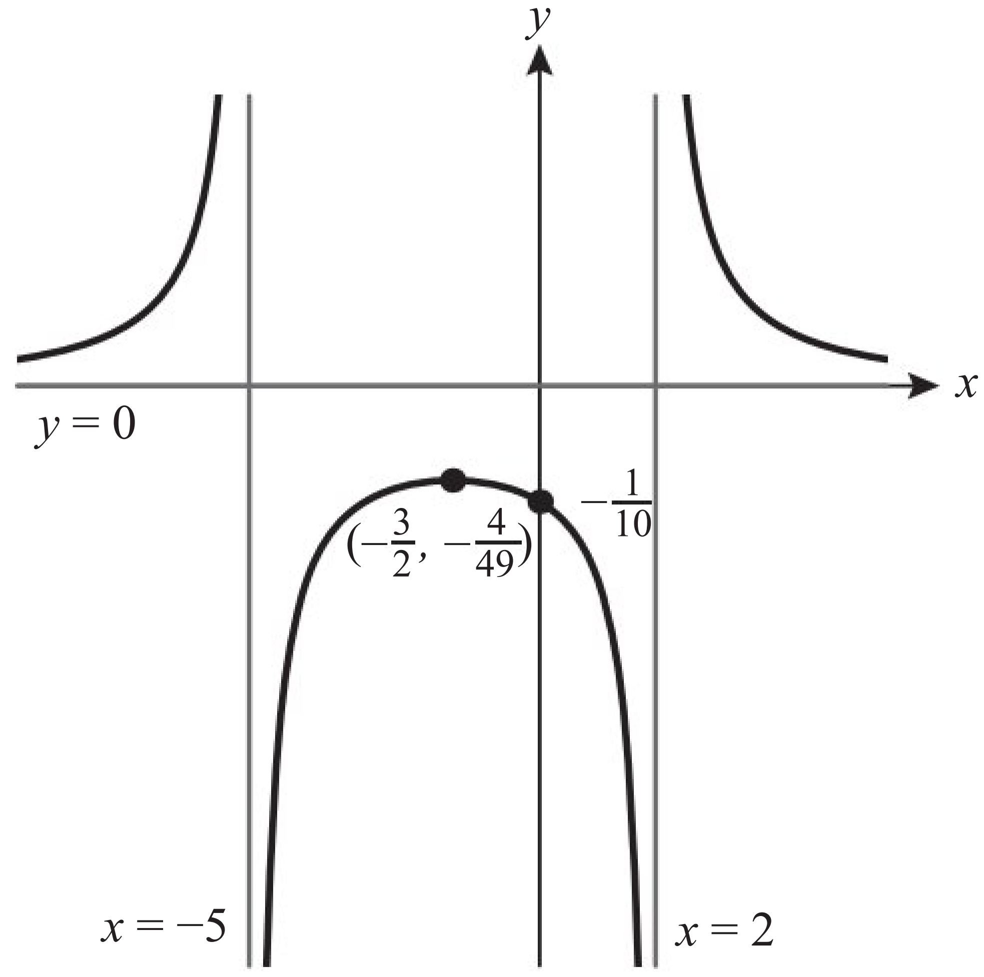
Correct intercepts and turning point labelled A1



ii Vertical asymptotes at x = −5, 2, y-int −0.1 A1

Parts of curve in correct quadrants M1

Turning point $\left(-\frac{3}{2}, -\frac{4}{49}\right)$ A1



[5 marks]
Total [20 marks]

11 a $\frac{dy}{dx} = -e^{-x}\sin 2x + 2e^{-x}\cos 2x$ M1A1

M1 for attempt at product rule or quotient rule

$-e^{-x}\sin 2x + 2e^{-x}\cos 2x = 0$

$-\sin 2x + 2\cos 2x = 0$ M1

$\frac{\sin 2x}{\cos 2x} = 2$ A1

$\tan 2x = 2$ AG

[4 marks]

b $\frac{d^2y}{dx^2} = e^{-x}\sin 2x - 2e^{-x}\cos 2x - 2e^{-x}\cos 2x - 4e^{-x}\sin 2x$ (M1)

M1 for attempt at product rule or quotient rule on their $\frac{dy}{dx}$

$= -3e^{-x}\sin 2x - 4e^{-x}\cos 2x$ A1

$-3e^{-x}\sin 2x - 4e^{-x}\cos 2x = 0$ M1

$\frac{\sin 2x}{\cos 2x} = -\frac{4}{3}$ A1

$\tan 2x = -\frac{4}{3}$ AG

[4 marks]

c x-intercept $= \frac{\pi}{2}$ A1

$\int e^{-x}\sin 2x\,dx = -e^{-x}\sin 2x - \int -2e^{-x}\cos 2x\,dx$ (M1)

$= -e^{-x}\sin 2x + 2\int e^{-x}\cos 2x\,dx$ A1

$= -e^{-x}\sin 2x + 2(-e^{-x}\cos 2x - \int 2e^{-x}\sin 2x\,dx)$

$= -e^{-x}\sin 2x - 2e^{-x}\cos 2x - 4\int e^{-x}\sin 2x\,dx$ A1

$5\int e^{-x}\sin 2x\,dx = -e^{-x}\sin 2x - 2e^{-x}\cos 2x$ (M1)

$\int_0^{\frac{\pi}{2}} e^{-x}\sin 2x\,dx = \frac{1}{5}[-e^{-x}\sin 2x - 2e^{-x}\cos 2x]_0^{\frac{\pi}{2}}$

$= \frac{1}{5}\left(2e^{-\frac{\pi}{2}} + 2\right)$

$= \frac{2}{5} + \frac{2}{5}e^{-\frac{\pi}{2}}$ A1A1A1

[8 marks]
Total [16 marks]

12	**a**	For any positive integer n, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$	A1
		True when $n = 1$: $(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$	A1
		Assume it is true for $n = k$:	
		$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$	M1
		Then	
		$(\cos\theta + i\sin\theta)^{k+1} = (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$	
		$= \cos(k+1)\theta + i\sin(k+1)\theta$	A1
		The statement is true for $n = 1$ and if it is true for some $n = k$ then it is also true for $n = k + 1$; it is therefore true for all integers $n > 1$ [by the principle of mathematical induction].	R1
			[5 marks]
	b	[Writing $c = \cos\theta$, $s = \sin\theta$:]	
		$(\cos\theta + i\sin\theta)^5 = c^5 + 5ic^4s - 10c^3s^2 - 10ic^2s^3 + 5cs^4 + is^5$	A1
		Equating real parts of $\cos 5\theta + i\sin 5\theta$ and $(\cos\theta + i\sin\theta)^5$:	
		$\cos 5\theta = c^5 - 10c^3s^2 + 5cs^4$	M1
		Using $s^2 = 1 - c^2$	
		$\cos 5\theta = c^5 - 10c^3(1 - c^2) + 5c(1 - c^2)^2$	(M1)
		$= c^5 - 10c^3 + 10c^5 + 5c(1 - 2c^2 + c^4)$	A1
		$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$	AG
			[4 marks]
	c	$5\theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \frac{9\pi}{10}$	M1
		Obtain at least $\theta = \frac{\pi}{10}$	A1
			[2 marks]
	d	The roots of the equation are cos(values above)	(M1)
		Either $c = 0$, in which case $\theta = \frac{\pi}{2}\ldots$	A1
		… or $16c^4 - 20c^2 + 5 = 0$	A1
		$c^2 = \frac{5 \pm \sqrt{5}}{8}$	A1
		$\cos\left(\frac{\pi}{10}\right)$ is positive and the largest of the roots	R1
		So $\cos\left(\frac{\pi}{10}\right) = \sqrt{\frac{5+\sqrt{5}}{8}}$	A1
			[6 marks]
	e	[$\cos\left(\frac{7\pi}{10}\right)$ is negative and not equal to $-\cos\left(\frac{\pi}{10}\right)$]	
		$\cos\left(\frac{\pi}{10}\right)\cos\left(\frac{7\pi}{10}\right) = \left(\sqrt{\frac{5+\sqrt{5}}{8}}\right)\left(-\sqrt{\frac{5-\sqrt{5}}{8}}\right)$	M1
		$\left[= -\sqrt{\frac{25-5}{64}}\right] = -\frac{\sqrt{5}}{4}$	A1
			[2 marks]
			Total [19 marks]

Practice Set A: Paper 2 Mark scheme

SECTION A

1 **a** $= \frac{4}{3}\pi(3^3) \times 1.45$ (M1)

$= 164\,\text{g}$ A1

b Each volume [mass] is $\frac{1}{8}$ the previous one. A1

Sum to infinity $= \frac{164}{1-\frac{1}{8}} = 187\,\text{g}$ M1A1

Hence the mass is always smaller than 200 g. A1

[6 marks]

2 $E(X) = \int_{2\pi}^{3\pi} 0.4106x\sin x\sqrt{x-2\pi}\,dx [= 8.018]$ M1

A1 for correct limits A1

$E(X^2) = \int_{2\pi}^{3\pi} 0.4106x^2\sin x\sqrt{x-2\pi}\,dx [= 64.71]$ M1

$\text{Var}(X) = 64.71 - 8.018^2$ M1

$= 0.425$ (A1)

$\sqrt{0.425} = 0.652$ (A1)

[6 marks]

3 Attempt sine rule:

$\frac{\sin\theta}{x-1} = \frac{\sin 2\theta}{x+2}$ A1

Use double angle formula:

$= \frac{2\sin\theta\cos\theta}{x+2}$ M1

$2\cos\theta(x-1) = x+2$ A1

Rearrange for x:

$x(2\,\text{Cos}\theta - 1) = 2 + 2\cos x$ M1

$x = \frac{2(1+\cos\theta)}{2\cos\theta - 1}$ A1

Use $\cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1$:

$= \frac{4\cos^2\left(\frac{\theta}{2}\right)}{4\cos^2\left(\frac{\theta}{2}\right) - 3}$ M1

Divide by $\cos^2\left(\frac{\theta}{2}\right)$, clearly using $= \frac{1}{\cos\left(\frac{\theta}{2}\right)} = \sec\left(\frac{\theta}{2}\right)$:

$= \frac{4}{4 - 3\sec^2\left(\frac{\theta}{2}\right)}$ A1AG

[7 marks]

4 **a** A A1

Gradient is zero and changing from positive to negative R1

b B, D A1

and E A1

Second derivative is zero and changes sign R1

[5 marks]

5 $(4-x)^{-\frac{1}{2}}$ M1

$=4^{-\frac{1}{2}}\left(1-\frac{x}{4}\right)^{-\frac{1}{2}}$ M1

$\approx \frac{1}{2}(1+\frac{x}{8}+\ldots$ A1

$\ldots+\frac{3}{8}\left(-\frac{x}{4}\right)^2-\frac{5}{16}\left(-\frac{x}{4}\right)^3)$ M1

$=\frac{1}{2}+\frac{x}{16}+\frac{3x^2}{256}+\frac{5x^3}{2048}$ A1

Valid for $|x|<4$ A1

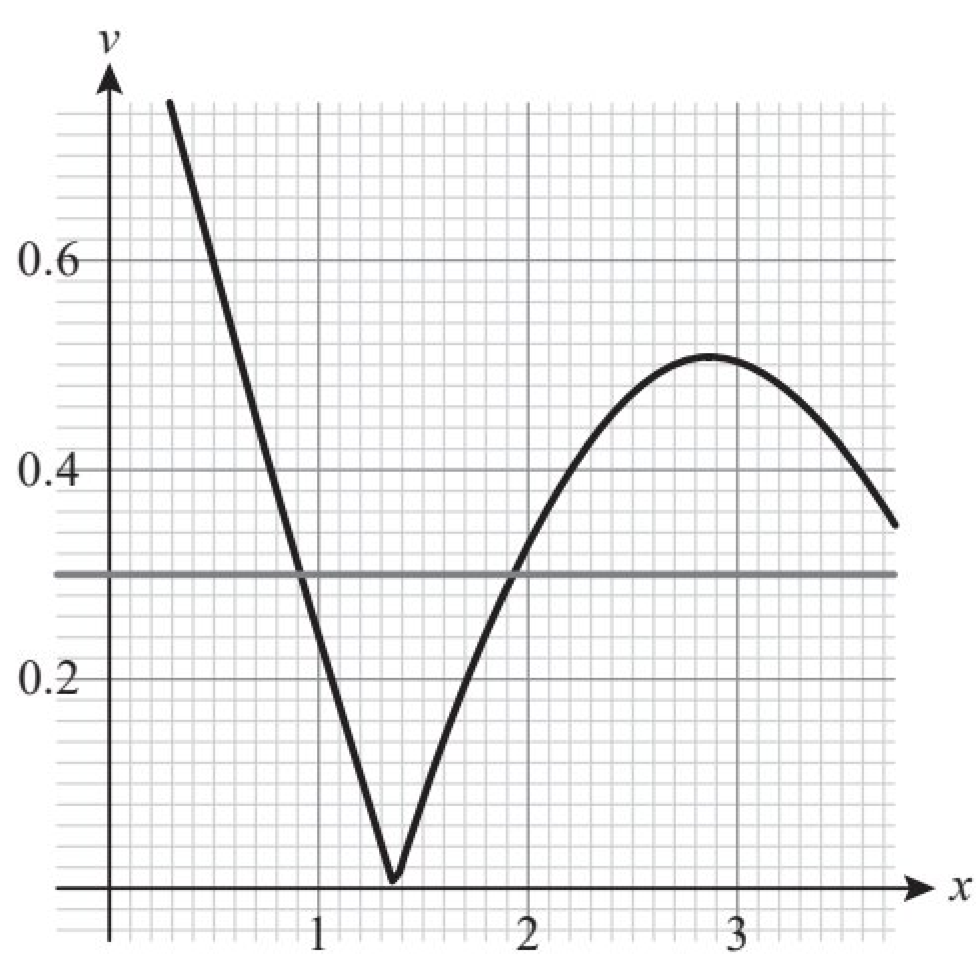
[6 marks]

6 **a** integrate $|v|$ (M1)

With limits 0 and 5 (M1)

Distance = 1.8 m A1

b Sketch $\left|\frac{dv}{dt}\right|$ [or $\frac{dv}{dt}$] (M1)



Intersect with $y = 0.3$ [or with both 0.3 and -0.3] (M1)

$t = 0.902$ and 1.93 seconds A1

[6 marks]

7 Consider f(1):

$1^2 - 1 = -(1)^2 + b(1) + c$ M1

$\Rightarrow b + c = +1$ A1

Consider f□(1):

$+2(1) = -2(1) + b$ M1A1

$\Rightarrow b = 4$ A1

$c = -3$ A1

[6 marks]

8 $|\mathbf{a}||\mathbf{b}|\cos\theta = 17$ M1

$|\mathbf{a}||\mathbf{b}|\sin\theta = \sqrt{4+1+25}$ [$=\sqrt{30}$] M1

$\tan\theta = \frac{\sqrt{30}}{17}$ M1A1

$\theta = 17.9°$ A1

[5 marks]

9	**a**		8! seen	A1
			8!2!2! = 161280	(M1)A1
	b		Pair 1 stands together: 9!2! [= 725760]	M1
			2 × 725760 –(their **a**) [=1 290 240]	M1
			$\frac{10! - 1\,290\,240}{10!}$	M1
			= 0.644 (3 s.f.)	A1
				[7 marks]

SECTION B

10	**a**		Paper 1: mean = 78.9, SD = 17.4	A1
			Paper 2: mean = 74.0, SD = 15.1	A1
			Paper 1 has higher marks on average.	A1
			Paper 2 has more consistent marks.	A1
				[4 marks]
	b		$r = 0.868$	A1
			> 0.532	M1
			There is evidence of positive correlation between the two sets of marks.	A1
				[3 marks]
	c	**i**	Find regression line x on y	M1
			$x = 0.997y + 5.16$	A1
			0.997 × 86 + 5.16 ≈ 91 marks	A1
		ii	Can't be used.	A1
			Mark is outside of the range of available data (interpolation)	R1
				[5 marks]
	d	**i**	Boundary for 7: inverse normal of 0.88	M1
			Boundary = 81	A1
			5 students	A1
		ii	Use $B(12, 0.12)$	(M1)
			P(>5) = 1 – P($\simeq$ 5)	(M1)
			= 0.001 44	A1
				[6 marks]
	e		Scaled mark = $\frac{80}{110}$ × original mark	(M1)
			Mean = 57.4	A1
			SD = 12.7	A1
				[3 marks]
				Total [21 marks]

11	**a**		Separate variables and attempt integration	M1
			$\int \frac{dy}{y} = \int \tan x\,dx$	A1
			$\ln y = -\ln \lvert\cos x\rvert + c$	A1
			$y = Ae^{-\ln(\cos x)}$	M1
			$= \frac{A}{\cos x}$	A1
				[5 marks]
	b	**i**	$\int -\tan x\,dx = \int \frac{-\sin x}{\cos x}dx = \ln(\cos x)$	M1A1
			$I = e^{\ln(\cos x)} = \cos x$	M1(AG)
		ii	$y\cos x = \int \cos^2 x\,dx$	M1
			$= \int \frac{\cos 2x + 1}{2}dx$	M1
			$= \frac{1}{4}\sin 2x + \frac{1}{2}x + c$	A1
			$y = \frac{\sin 2x}{4\cos x} + \frac{x}{2\cos x} + \frac{c}{\cos x}\ \left(= \frac{\sin x}{2} + \frac{x\sec x}{2} + c\sec x\right)$	A1
				[7 marks]

c Use $y_{n+1} = y_n + 0.1(y_n^2 \tan x_n + \cos x_n)$ M1A1
Table of values – at least the first two rows correct M1

x	*y′*	*y*
0	1.000	2.000
0.1	1.437	2.100
0.2	2.001	2.244
0.3	2.803	2.444
0.4	4.058	2.724
0.5	6.229	3.130

$y(0.5) = 3.13$ A1

[4 marks]

Total [16 marks]

12 a i Equate *x*, *y*, *z* components:

$$\begin{cases} 5+7\lambda = 1-\mu \,(1) \\ 3+2\lambda = -8+3\mu \,(2) \\ 1-3\lambda = -2+2\mu \,(3) \end{cases}$$ M1A1

From, e.g. (1) and (2): $\lambda = -1,\ \mu = 3$ A1A1
Check in (3):
$1 - 3(-1) = 4$
$-2 + 2(3) = 4$
So lines intersect. M1AG

ii Substitute their values of λ and μ into either equation

$$\mathbf{r} = \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 7 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$$ (M1)

So coordinates $(-2,\ 1,\ 4)$ A1

[7 marks]

b Attempt to find cross product of direction vectors:

$$\begin{pmatrix} 7 \\ 2 \\ -3 \end{pmatrix} \times \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$$ (M1)

$$= \begin{pmatrix} 13 \\ -11 \\ 23 \end{pmatrix}$$ A1

[2 marks]

c $\mathbf{r} \cdot$ their $\mathbf{n}$ = their $\mathbf{p} \cdot$ their $\mathbf{n}$

$$\mathbf{r} \cdot \begin{pmatrix} 13 \\ -11 \\ 23 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 13 \\ -11 \\ 23 \end{pmatrix}$$ (M1)

$$\mathbf{r} \cdot \begin{pmatrix} 13 \\ -11 \\ 23 \end{pmatrix} = 55$$ A1

[2 marks]

d $\overrightarrow{QP} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} -11 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 1 \\ 3 \end{pmatrix}$ (M1)

$\cos\phi = \dfrac{\begin{pmatrix} 9 \\ 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 13 \\ -11 \\ 23 \end{pmatrix}}{\sqrt{9^2+1^2+3^2}\sqrt{13^2+11^2+23^2}}$ M1A1

$= \dfrac{25}{39}$ A1

$\sin\theta = \sin(90-\phi) = \cos\phi$ M1

So, $\sin\theta = \dfrac{25}{39}$ A1

[6 marks]

e $d = \left|\overrightarrow{QP}\right| \sin\theta$ (M1)

$= \dfrac{25\sqrt{91}}{39}$ A1

[2 marks]

Total [19 marks]

Practice Set A: Paper 3 Mark scheme

1 **a** $F_3 = 2$ A1
$F_4 = 3$ A1
$F_5 = 5$ A1
[3 marks]

b $F_{12} = 144$ A1
This is another Fibonacci number which is a perfect square R1
[2 marks]

c Check that the statement is true for $n = 1$: M1
$LHS = 1^2 = 1$ $RHS = 1 \times 1 = 1$ A1
Assume true for $n = k$
$\sum_{i=1}^{i=k} (F_i)^2 = F_k F_{k+1}$ A1
Then
$\sum_{i=1}^{i=k+1} (F_i)^2 = \sum_{i=1}^{i=k} (F_i)^2 + (F_{k+1})^2$ M1
$= F_k F_{k+1} + (F_{k+1})^2$
$= F_{k+1}(F_k + F_{k+1})$
$= F_{k+1} F_{k+2}$ A1
So if the statement works for $n = k$ then it works for $n = k + 1$ and it works for $n = 1$ therefore it works for all $n \in \mathbb{Z}^+$. R1
[6 marks]

d Smallest such k is 5 A1
Check that the statement is true for $n = 5$ and $n = 6$: M1
$F_5 = 5$, $F_6 = 8$ A1
Assume true for $n = k$ and $n = k + 1$ M1
$F_k \geqslant k$, $F_{k+1} \geqslant k + 1$ A1
Then
$F_{k+2} = F_k + F_{k+1} \geqslant 2k + 1 > k + 2$ since $k > 1$ A1
So if the statement works for $n = k$ and $n = k + 1$ then it works for $n = k + 2$ and it works for $n = 5$ and $n = 6$ therefore it works for all integers $n \geqslant 5$ R1
[7 marks]

e $\alpha^{n+2} = \alpha^{n+1} + \alpha^n$ M1
Dividing by α^n since $\alpha \neq 0$: $\alpha^2 = \alpha + 1$ or $\alpha^2 - \alpha - 1 = 0$ A1
Using the quadratic formula $\alpha = \frac{1 \pm \sqrt{5}}{2}$ A1A1
[4 marks]

f $F_n + F_{n+1} = A\alpha_1^n + B\alpha_2^n + A\alpha_1^{n+1} + B\alpha_2^{n+1}$ M1
$A(\alpha_1^n + \alpha_1^{n+1}) + B(\alpha_2^n + \alpha_2^{n+1})$
$A\alpha_1^{n+2} + B\alpha_2^{n+2} = F_{n+2}$ A1
[2 marks]

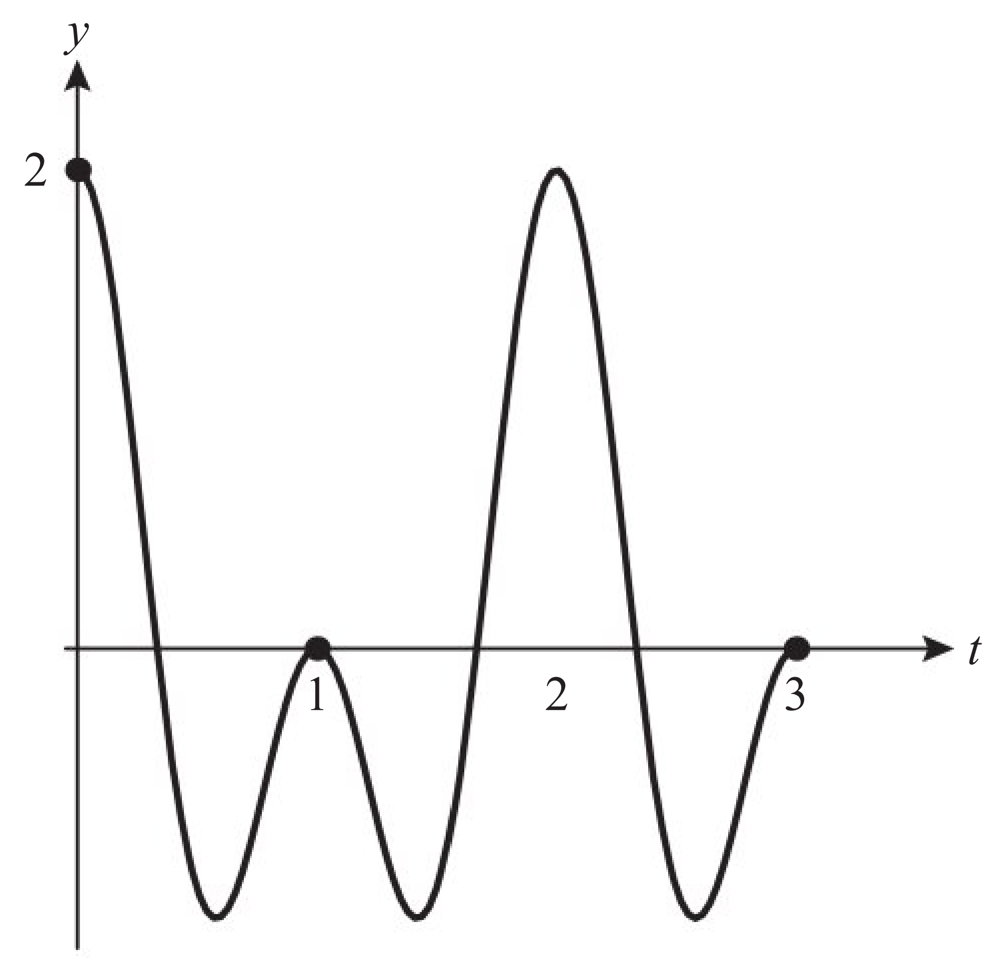
g $F_1 = A\alpha_1 + B\alpha_2 = 1$ A1
$F_2 = A\alpha_1^2 + B\alpha_2^2 = 1$ A1
Since $\alpha^2 = \alpha + 1$:
$A(\alpha_1 + 1) + B(\alpha_2 + 1) = 1$ M1
$A\alpha_1 + B\alpha_2 + A + B = 1$
$A + B = 0$
$A = -B$
Substituting into first equation: $A(\alpha_1 - \alpha_2) = 1$
$A = \frac{1}{\alpha_1 - \alpha_2} = \frac{1}{\sqrt{5}}$
$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$ A1
[4 marks]

h As n gets large, $\left(\frac{1-\sqrt{5}}{2}\right)^n \to 0$ M1
$\frac{F_{n+1}}{F_n} \approx \frac{\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}\right)}{\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n\right)} = \frac{1+\sqrt{5}}{2}$ A1
[2 marks]
Total [30 marks]

2 **a** 2 A1

[1 mark]

b **i**



A1

ii 2 A1

[2 marks]

c **i** $A = 4$ A1

$B = 8$ A1

$C = 20$ A1

ii $T = 2n$ A1

[4 marks]

d $f(t + 2n) = \cos\left(\pi\left(t + 2n\right)\right) + \cos\left(\pi\left(1 + \frac{1}{n}\right)\left(t + 2n\right)\right)$ M1

$= \cos\left(\pi t + 2n\pi\right) + \cos\left(\left(1 + \frac{1}{n}\right)\pi t + 2\pi\left(n + 1\right)\right)$ A1

$= \cos\left(\pi t\right) + \cos\left(\left(1 + \frac{1}{n}\right)\pi t\right) = f\left(t\right)$

Since $\cos(x + 2\pi k) = \cos x$ if k is an integer R1

[3 marks]

e **i** $\cos(A + B) + \cos(A - B)$

$= \cos A \cos B - \sin A \sin B + \cos A \cos B + \sin A \sin B$

$= 2 \cos A \cos B$ A1

ii If $P = A + B$ and $Q = A - B$ then

$A = \frac{P + Q}{2}, B = \frac{P - Q}{2}$ M1

$\cos P + \cos Q = 2 \cos\left(\frac{P + Q}{2}\right) \cos\left(\frac{P - Q}{2}\right)$ A1

[3 marks]

f $f(t) = 2 \cos\left(\pi\left(1 + \frac{1}{2n}\right)t\right) \cos\left(\frac{\pi}{2n} t\right)$ A1

The graph of $\cos \left(\pi \left(1 + \frac{1}{2n}\right)t\right)$ provides the high frequency oscillations. R1

Their amplitude is determined/enveloped by the lower frequency curve $\cos\left(\frac{\pi}{2n} t\right)$ R1

[3 marks]

g $\frac{d^2 x}{dt^2} = -\omega^2 \cos \omega t$ M1A1

The DE becomes:

$-\omega^2 \cos \omega t + 4 \cos \omega t = 0$ M1

This is solved when $\omega^2 = 4$ so $\omega = 2$ A1

[4 marks]

h $\frac{d^2x}{dt^2} = -4 \cos 2t - k^2 g(k) \cos kt$ M1

The DE becomes:

$-4 \cos 2t - k^2 g(k) \cos kt + 4 \cos 2t + 4g(k) \cos kt = \cos kt$ M1

$(4g(k) - k^2 g(k)) \cos kt = \cos kt$

This is true for all t when $g(k)(4 - k^2) = 1$

$g(k) = \frac{1}{4 - k^2}$ A1

[3 marks]

i When $k = 2$ A1

Since $\frac{1}{4 - k^2} \to \infty$ as $k \to 2$ R1

[2 marks]

Total [25 marks]

Practice Set B: Paper 1 Mark scheme

SECTION A

1 $k\ln(x^2+3)$ M1

$2\ln(x^2+3)$ A1

Limits: $2\ln(a^2+3) - 2\ln 3$ M1

$2\ln\left(\frac{a^2+3}{3}\right) = \ln 16$ or $2\ln(a^2+3) = \ln(16\times 9)$ A1

$\left(\frac{a^2+3}{3}\right)^2 = 16$ or $(a^2+3)^2 = 16\times 9$ M1

$\frac{a^2+3}{3} = 4$ or $a^2+3 = 12$ only A1

$a = 3$ A1

[7 marks]

2 **a** $\frac{1}{4}$ or 10 seen A1

$\frac{10}{40}\times\frac{9}{39}$ (M1)

$=\frac{3}{52}$ A1

b $\frac{10}{40}\times\frac{20}{39}$ (M1)

$\times 2$ (M1)

$=\frac{10}{39}$ A1

[6 marks]

3 Attempt quotient rule:

$\frac{dy}{dx} = \frac{x\cos x - \sin x}{x^2}$ M1A1

$=\frac{-\pi - 0}{\pi^2} = -\frac{1}{\pi}$ A1

$y = 0$ A1

$y = \pi(x-\pi)$ A1

[5 marks]

4 Sketch both graphs or consider four cases M1

Attempts to find intersection points: M1

$x - 3 = 2x + 1 \Rightarrow x = -4$ A1

$-x + 3 = 2x + 1 \Rightarrow x = \frac{2}{3}$ A1

$x \leqslant -4$ or $x \geqslant \frac{2}{3}$ A1

[5 marks]

5 $P(B|A) = \frac{P(A\cap B)}{P(A)} : 0.6 = \frac{P(A\cap B)}{0.3}$ (M1)

$P(A\cap B) = 0.18$ A1

$P(A\cup B) = P(A) + P(B) - P(A\cap B) : 0.8 = 0.3 + P(B) - 0.18$ M1

$P(B) = 0.68$ A1

$P(B|A) = \frac{P(A\cap B)}{P(B)}\left[=\frac{0.18}{0.68}\right]$ M1ft

$=\frac{9}{34}$ A1

[6 marks]

6	$A = -1$	A1
	$x = 0$: $A + B = 8$	M1
	$\Rightarrow B = 9$	A1
	$-1 + 9e^{-2k} = 0 \Rightarrow e^{-2k} = \frac{1}{9}$	M1
	Attempt taking logarithm of both sides, e.g. $2k = -\ln\left(\frac{1}{9}\right)$	M1
	$k = \ln 3$	A1
		[6 marks]
7	$7^1 + 3^0 = 8$, so true for $n = 1$	A1
	Assume that, for some k, $7^k + 3^{k-1} = 4A$	M1
	Then	
	$7^{k+1} + 3^k = 7 \times 7^k + 3 \times (4A - 7^k)$	M1
	$= 4 \times 7^k + 12A$	A1
	So $7^{k+1} + 3^k$ is divisible by 4	A1
	The statement is true for $n = 1$, and if it is true for some $n = k$ then it is also true for $n = k + 1$. Therefore it is true for all integers $n \geqslant 1$ [by the principle of mathematical induction].	A1
		[6 marks]
8	$\left\|4 - 4\sqrt{3}\,i\right\| = \sqrt{16 + 48} = 8$	M1
	$\Rightarrow \|z\| = 2$	A1
	$\arg(4 - 4\sqrt{3}\,i) = \arctan\left(-\sqrt{3}\right)$	M1
	$= -\frac{\pi}{3}\left[\text{or } \frac{5\pi}{3}\right]$	A1
	$\Rightarrow \arg z = -\frac{\pi}{9}, \ldots$	M1
	$\ldots \frac{5\pi}{9}$ or $\frac{11\pi}{9}$	A1
	$\therefore z = 2e^{-\frac{\pi}{9}i}, 2e^{\frac{5\pi}{9}i}, 2e^{\frac{11\pi}{9}i}, \left[\text{or } 2e^{\frac{5\pi}{9}i}, 2e^{\frac{11\pi}{9}i}, 2e^{\frac{17\pi}{9}i}\right]$	A1
		[7 marks]
9	$\cos x \approx 1 - \frac{x^2}{2} + \frac{x^4}{24}$	A1
	$\left(1 - x^2\right)^{-\frac{1}{2}}$	M1
	$\approx 1 + \frac{x^2}{2}$	A1
	$+\frac{3x^4}{8}$	A1
	Multiply the two expansions, using at least two terms in each one	M1
	Attempt to simplify, obtain at least $1 + 0x^2$	A1
	$1 + \frac{x^4}{6}$	A1
		[7 marks]

SECTION B

10 a	Use product rule	
	$f'(x) = e^{-kx} + x(-k)e^{-kx}$	M1
	$= (1 - kx)e^{-kx}$	A1AG
	Use product rule again, $u' = -k$, $v' = -ke^{-kx}$	M1
	$f''(x) = (-k)e^{-kx} + (1 - kx)(-k)e^{-kx}$	A1
	$= (k^2x - 2k)e^{-kx}$	A1
		[5 marks]

b $f'(x) = 0: (1 - kx)e^{-kx} = 0$ M1

$e^{-kx} \neq 0$ A1

$x = \frac{1}{k}$ A1

$f''\left(\frac{1}{k}\right) = \left(\frac{k^2}{k} - 2k\right)e^{-\frac{k}{k}}$ M1

$= -ke^{-1} < 0 \therefore$ local maximum A1

[5 marks]

c $f''(x) = 0: k^2x - 2k = 0$ M1

$x = \frac{2}{k}$ A1

The coordinates are

$\left(\frac{2}{k}, \frac{2}{k}e^{-2}\right)$ A1

[3 marks]

Integration by parts:

$\int_{\frac{1}{k}}^{\frac{2}{k}} x\,e^{-kx}\,dx = \left[-\frac{x}{k}e^{-kx}\right]_{\frac{1}{k}}^{\frac{2}{k}} + \int_{\frac{1}{k}}^{\frac{2}{k}} \frac{1}{k}e^{-kx}\,dx$ M1A1

$= \left[-\frac{2}{k^2}e^{-2} + \frac{1}{k^2}e^{-1}\right] - \left[\frac{1}{k^2}e^{-kx}\right]_{\frac{1}{k}}^{\frac{2}{k}}$ A1

$= \frac{2}{k^2}e^{-1} - \frac{3}{k^2}e^{-2}$ A1

$= \frac{2}{k^2 e} - \frac{3}{k^2 e^2}$ A1

$= \frac{2e - 3}{k^2 e^2}$ AG

[5 marks]

Total [18 marks]

11 a Eliminate a variable between two equations, e.g. x between equations (2) and (3):

$$\begin{cases} 6x + ky + 2z = a \\ 6x - y - z = 7 \\ 2y - z = 1 \end{cases}$$ M1

Eliminate the same variable between another pair of equations, e.g. x between (1) and (2):

$$\begin{cases} 6x + ky + 2z = a \\ (k+1)y + 3z = a - 7 \\ 2y - z = 1 \end{cases}$$ M1

Eliminate a variable between the pair of equations in two variables, e.g. z between (2) and (3):

$$\begin{cases} 6x + ky + 2z = a \\ (k+1)y + 3z = a - 7 \\ (k+7)y = a - 4 \end{cases}$$ M1

Leading to:

$(2k + 14)x = a + 10 + 2k$

OR

$(k + 7)y = a - 4$

OR

$(k + 7)z = 2a - 15 - k$ A1

Their coefficient of $x/y/z = 0$ (M1)

$k = -7$ A1

[6 marks]

b **i** Their RHS = 0 (with their value of k) (M1)

$a = 4$ A1

ii Let $z = \lambda$ M1

$2y - z = 1$ (M1)

$6x - y - z = 7$ (M1)

At least one of

$y = \dfrac{1+\lambda}{2}$, $x = \dfrac{5+\lambda}{4}$ A1ft

$$\mathbf{r} = \begin{pmatrix} \frac{5}{4} \\ \frac{1}{2} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$ A1

[7 marks]

c Normal vectors to each plane are

$$\begin{pmatrix} 6 \\ -7 \\ 2 \end{pmatrix}, \begin{pmatrix} 6 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

Since none of these are multiples of each other, no two planes are parallel M1

So the planes form a triangular prism A1

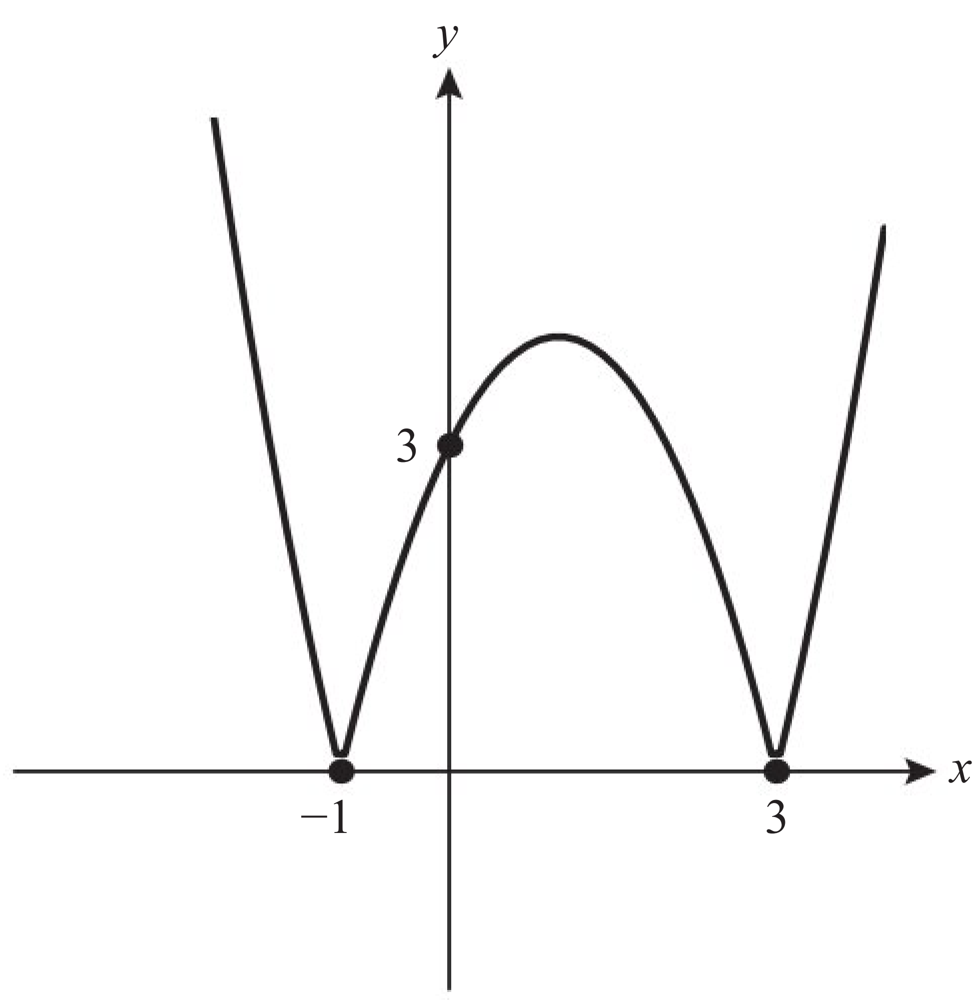
[2 marks]

Total [15 marks]

12 **a** Factorize to find x-intercepts: $(x - 3)(x + 1)$ M1

$(-1, 0)$ and $(3, 0)$ A1

Correct shape – reflected above x-axis A1



[3 marks]

b Solve $f(x) = -\frac{1}{2}x + 4$

$x^2 - 2x - 3 = -\frac{1}{2}x + 4$ (M1)

$2x^2 - 3x - 14 = 0$

$(2x - 7)(x + 2) = 0$

$x = \frac{7}{2}, -2$ A1

Solve $-f(x) = -\frac{1}{2}x + 4$

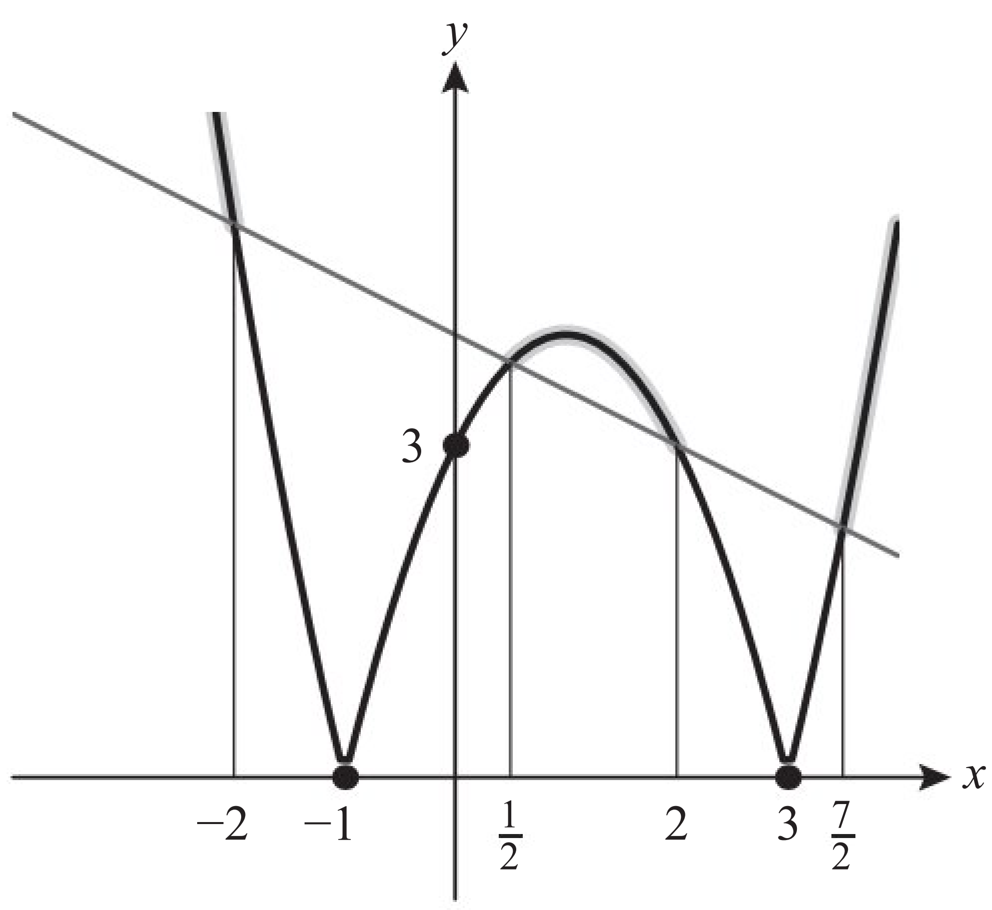
$-(x^2 - 2x - 3) = -\frac{1}{2}x + 4$ (M1)

$2x^2 - 5x + 2 = 0$

$(2x - 1)(x - 2) = 0$

$x = \frac{1}{2}, 2$ A1

Sketch of $y = |f(x)|$ and $y = -\frac{1}{2}x + 4$



$x < -2$ or $\frac{1}{2} < x < 2$ or $x > \frac{7}{2}$

Note: Award A1 for two correct regions A1A1

[6 marks]

c $x \in \mathbb{R}, x \neq -1, x \neq 3$ A1

[1 mark]

d $g'(x) = \dfrac{2(x^2 - 2x - 3) - (2x - 7)(2x - 2)}{(x^2 - 2x - 3)^2}$

Note: Award M1 for attempt at quotient rule M1A1

For turning points, $g'(x) = 0$:

$2(x^2 - 2x - 3) - (2x - 7)(2x - 2) = 0$ M1

$x^2 - 7x + 10 = 0$

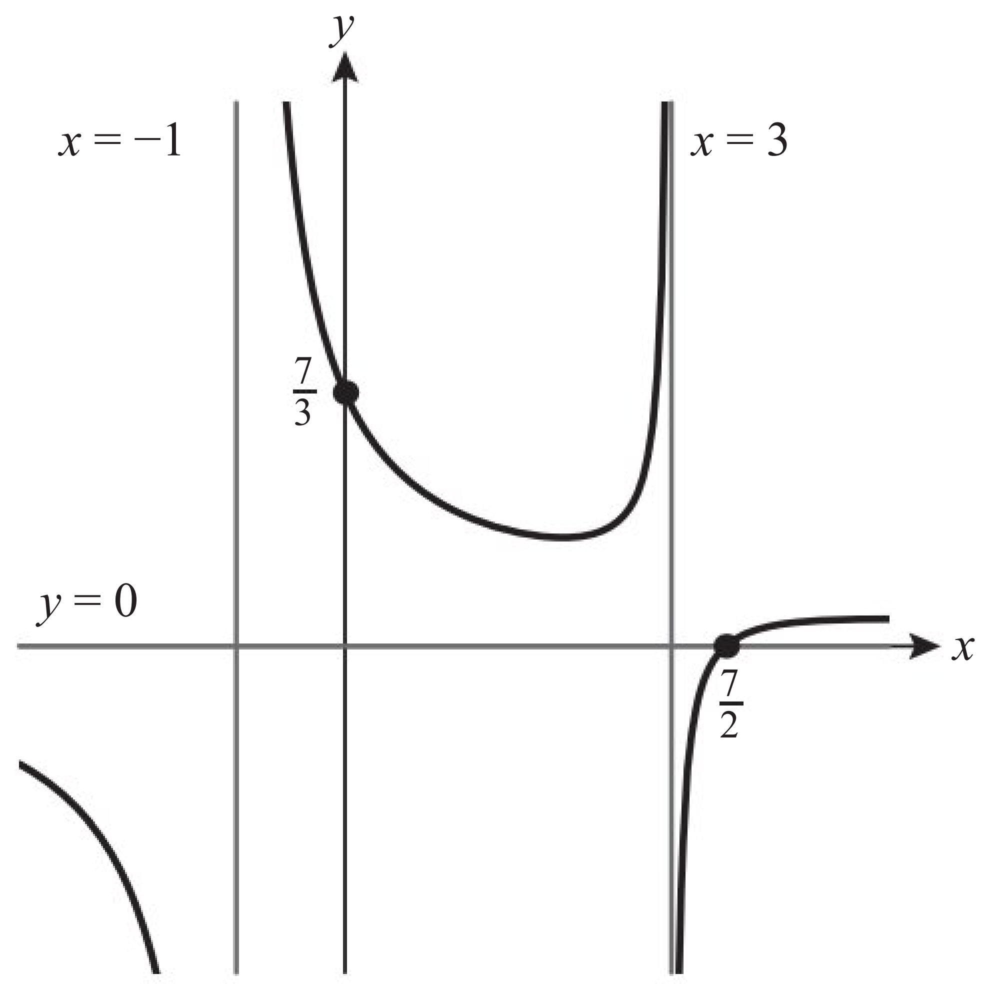
$(x - 2)(x - 5) = 0$

$x = 2, 5$ A1

So, coordinates (2, 1) and $\left(5, \frac{1}{4}\right)$ A1

[5 marks]

e



Correct shape between vertical asymptotes A1
Correct shape outside vertical asymptotes A1
Vertical asymptotes: $x = -1$, $x = 3$ A1
Horizontal asymptote: $y = 0$ A1
Axis intercepts at $\left(\frac{7}{2}, 0\right)$ and $\left(0, \frac{7}{3}\right)$ A1

[5 marks]

f $g(x) \in \left(-\infty, \frac{1}{4}\right] \cup [1, \infty)$ A2

[2 marks]

Total [22 marks]

Practice Set B: Paper 2 Mark scheme

SECTION A

1	$a + 4d = 7$, $a + 9d = 81$	M1A1
	Solving:	
	$a = -52.2$, $d = 14.8$	A1
	$S_{20} = \frac{20}{2}(-104.4 + 19 \times 14.8)$	(M1)
	$= 1768$	A1
		[5 marks]
2	Find the diagonal of the square base: $\sqrt{8.3^2 + 8.3^2}$	M1
	Height $= \frac{\sqrt{8.3^2 + 8.3^2}}{2}\tan(89.8°)$	M1
	$= 1681$	A1
	$= 1.7 \times 10^3$ cm	A1
		[4 marks]
3	mean = 131.9, SD = 7.41	A1
	Boundaries for outliers: mean ± SD	(M1)
	= 117.1, 146.7	A1A1ft
	147 is an outlier	A1
		[5 marks]
4	At least one correct use of compound angle formula	M1
	Correct values of $\sin\left(\frac{\pi}{3}\right)$ and $\cos\left(\frac{\pi}{3}\right)$ used	A1
	LHS $\equiv \dfrac{\left(\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x\right) - \left(\frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x\right)}{\left(\frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x\right) - \left(\frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x\right)}$	A1
	$\equiv \dfrac{\sqrt{3}\cos x}{-\sqrt{3}\sin x}$	A1
	$\equiv -\cot x$	A1(AG)
		[5 marks]
5	$\frac{2}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$	M1
	$2 = A(x-1) + Bx$	M1
	Using $x = 0$: $A = -2$	
	Using $x = 1$: $B = 2$ (both correct)	A1
	$\int -\frac{2}{x} + \frac{2}{x-1}\,dx = -2\ln x + 2\ln(x-1) + c$	M1A1
	$= \ln\left(\frac{x-1}{x}\right)^2 + c$	A1
		[6 marks]
6	gradient = 3.024	(M1)
	normal gradient $= -\frac{1}{\text{their gradient}}$ [−0.3307]	(M1)
	y-coordinate = 3.392	A1
	Equation of normal: $y - 3.392 = -0.3307(x - 1.5)$	A1
	A: $y = 0$, B: $x = 0$ [$x_A = 11.76$, $y_B = 3.888$]	(M1)
	Area = 22.9	A1
		[6 marks]
7	Differentiate implicitly: at least one term containing y correct	M1
	$6x + 2xy' + 2y - 2yy' = 0$	A1
	$y' = 0 \Rightarrow y = -3x$	M1
	Substitutes their expression for x or y back into curve:	
	$3x^2 + 2x(-3x) - (-3x)^2 + 24 = 0$	M1A1
	$12x^2 = 24 \Rightarrow x = \pm\sqrt{2}$	A1
	$(\sqrt{2}, -3\sqrt{2})$, $(-\sqrt{2}, 3\sqrt{2})$	A1
		[7 marks]

8 **a**	Limits $\sqrt[3]{5}, \sqrt[3]{17}$ (seen in either part)	A1
	$x = \sqrt{y^3 - 1}$	A1
	$\int \sqrt{y^3 - 1}\, dy$	M1
	$= 2.57$	A1
b	Using x^2	M1
	$\int \pi (y^3 - 1)\, dy$	M1
	$= 24.9$	A1
		[7 marks]
9	Another root is $2 + i$	A1
	Consider sum of roots:	
	$(2 + i) + (2 - i) + x_3 = 7$ (allow -7)	M1
	$x_3 = 3$	A1
	Product of roots: $3(2 + i)(2 - i)$	M1
	$c = -15$	A1
		[5 marks]
10	The rth term is	
	$nC_r x^{2r} \left(\frac{1}{x}\right)^{n-r}$	(M1)
	For constant term: $2r - (n - r) = 0$	(M1)
	$n = 3r$	A1
	So need $(3r)C_r = 495$	(M1)
	Using GDC: $r = 4$ so $n = 12$	A1
		[5 marks]

SECTION B

11 **a** **i**	$\frac{72 - \mu}{\sigma} = 0.8416$	M1
	$72 - \mu = 0.8416\sigma$	A1
	$\mu + 0.8416\sigma = 72$	(AG)
	$\frac{24 - \mu}{\sigma} = \ldots$	M1
	$\ldots -1.645$	A1
	$\mu - 1.645\sigma = 24$	A1
ii	(From GDC) $\mu = 55.8$, $\sigma = 19.3$	A1
	$P(>48) = 0.657$	A1
		[7 marks]
b	Use inverse normal with $p = 0.25$ or $p = 0.75$	
	($Q_1 = 42.8$ or $Q_3 = 68.8$)	M1
	IQR = 26 (hours)	A1
		[2 marks]
c	Use $B(20, 0.656)$	(M1)
	$1 - P(\leqslant 9)$	(M1)
	$= 0.953$	A1
		[3 marks]
d	$\frac{P(>72)}{P(>48)}$	(M1)
	$= 0.305$	A1
		[2 marks]
e	P(keep phone) $= 1 - (0.05 \times 0.9 + 0.75 \times 0.2)$	(M1M1)
	$\frac{0.2}{\text{P(keep phone)}}$	M1
	$= 0.248$	A1
		[4 marks]
		Total [18 marks]
12 **a**	$\cos\theta = \frac{2^2 + 4^2 - 4^2}{2 \times 2 \times 4}$ [= 0.25]	(M1)
	$\sin\theta = \sqrt{\frac{15}{16}}$ [= 0.968]	M1
	Area $= \frac{1}{2}(2 \times 4) \times$ their $\sin\theta$	M1
	$= 3.87$ [cm^2]	A1
		[4 marks]

b The third side is $10 - 3x$ … M1

… which must be positive. A1

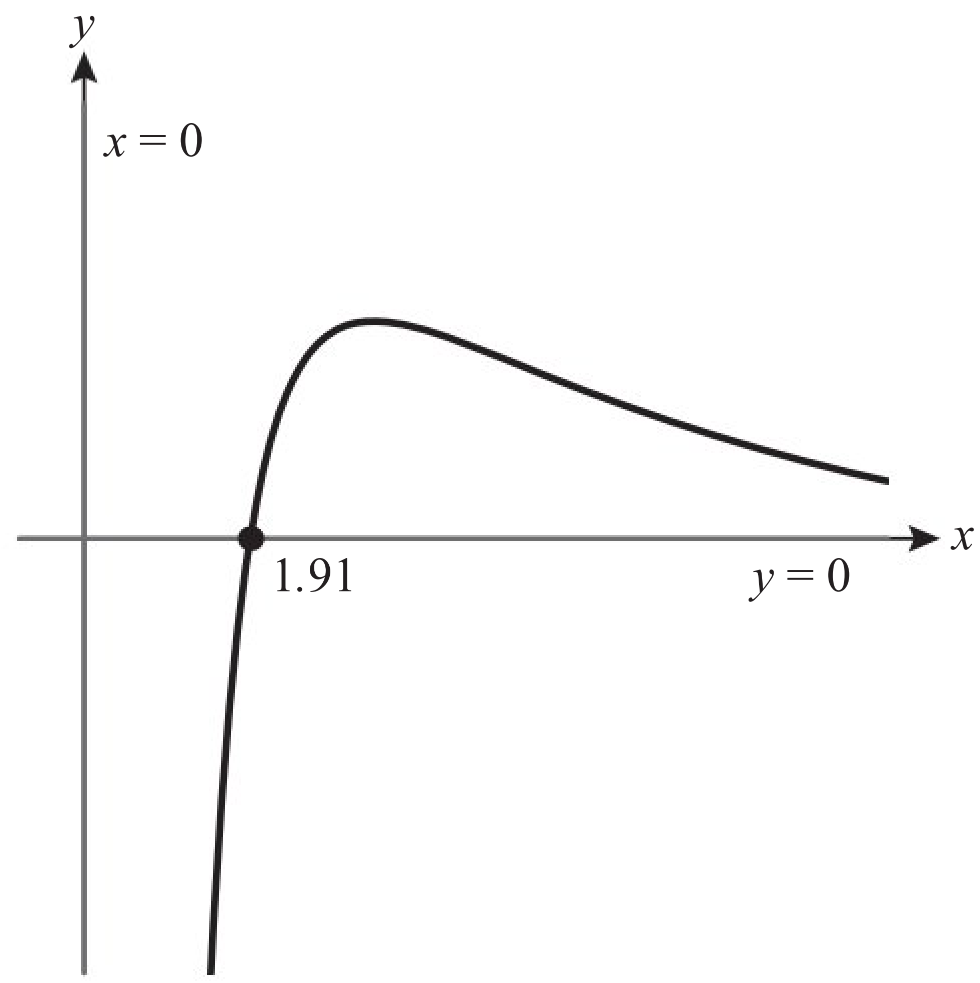
[2 marks]

c **i** $(10 - 3x)^2 = x^2 + (2x)^2 - 2x(2x)\cos\theta$ M1

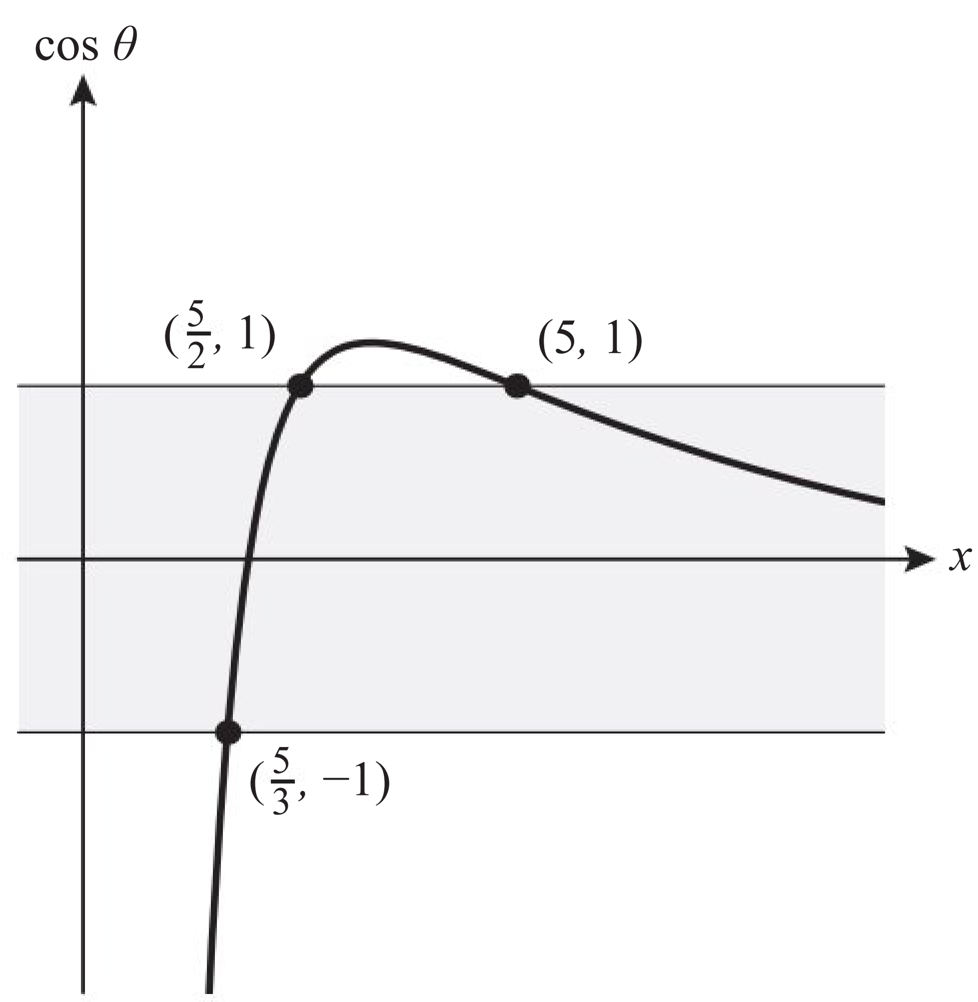
$$\cos\theta = \frac{60x - 4x^2 - 100}{4x^2}$$

$$= \frac{15x - x^2 - 15}{x^2}$$ A1(AG)

ii A2



iii Need $-1 < \cos\theta < 1$ (allow $\leqslant$ here) M1



Intersections at $x = \frac{5}{3}, \frac{5}{2}, 5$ A1

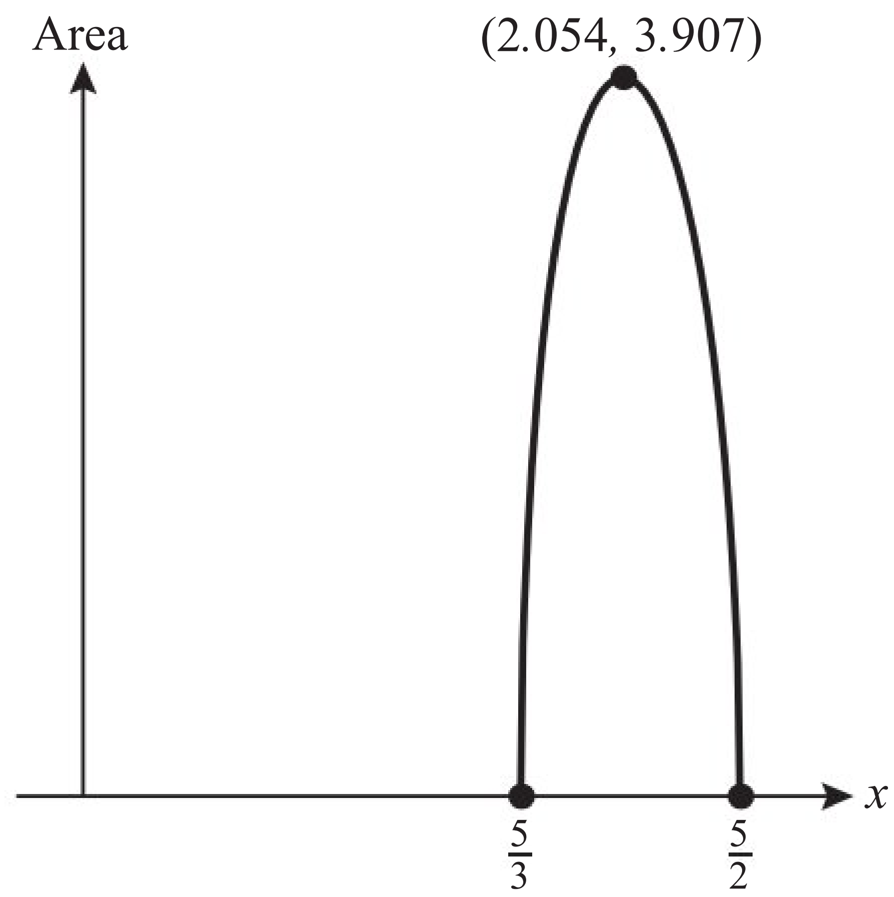
So $\frac{5}{3} < x < \frac{5}{2}$ A1

[7 marks]

d State or use $\sin\theta = \sqrt{1 - \cos^2\theta}$ M1

State or use Area $= \frac{1}{2}x(2x)\sin\theta$ M1

Sketch area as a function of x: M1



Max area for $x = 2.05$ A1

Max area = 3.91 [cm^2] A1

[5 marks]

Total [18 marks]

13 a Use quotient rule M1

Use implicit differentiation M1

$$\frac{d^2y}{dx^2} = \frac{\frac{dy}{dx}(x+y) - y\left(1+\frac{dy}{dx}\right)}{(x+y)^2}$$ A1

$$= \frac{x\frac{dy}{dx} - y}{(x+y)^2}$$ A1

Substitute $\frac{dy}{dx} = \frac{y}{x+y}$:

$$\frac{d^2y}{dx^2} = \frac{\frac{xy}{x+y} - y}{(x+y)^2}$$ M1

$$= \frac{xy - y(x+y)}{(x+y)^3}$$ M1

$$-\frac{y^2}{(x+y)^3}$$ A1

[7 marks]

b $\frac{dy}{dx} = v + x\frac{dv}{dx}$ M1

$v + x\frac{dv}{dx} = \frac{xv}{x+xv}$ M1

$x\frac{dv}{dx} = \frac{v}{1+v} - v$ M1

$= -\frac{v^2}{1+v}$ A1

[4 marks]

c Separate variables: $\frac{1+v}{v^2}\frac{dv}{dx} = -\frac{1}{x}$ or equivalent M1

$\int \frac{1+v}{v^2}\,dv = \int -\frac{1}{x}\,dx$ M1

$-\frac{1}{v} + \ln v = -\ln x + c$ A1

Using $x = 1, y = 1, v = 1$: $-1 + 0 = 0 + c$ M1

$c = -1$ A1

$-\frac{1}{v} + \ln(xv) = -1$ (M1)

$\frac{x}{y} = \ln y + 1$ (M1)

$x = y(\ln y + 1)$ A1

[8 marks]

Total [19 marks]

Practice Set B: Paper 3 Mark scheme

1 **a** Check that the statement is true for $n = 1$: M1

LHS = 1 RHS = $\frac{1 \times 2}{2} = 1$ A1

Assume true for $n = k$ M1

$\sum_{r=1}^{r=k} r = \frac{k(k+1)}{2}$ A1

Then

$\sum_{r=1}^{r=k+1} r = \sum_{r=1}^{r=k} r + (k+1) = \frac{k(k+1)}{2} + (k+1)$ M1

$= (k+1)\left(\frac{k}{2} + 1\right)$

$= \frac{(k+1)(k+2)}{2}$ A1

So if the statement works for $n = k$ then it works for $n = k + 1$ and it works for $n = 1$, therefore it works for all $n \in \mathbb{Z}^+$. A1

[7 marks]

b $3n^2 + 3n + 1$ M1A1

[2 marks]

c $\sum_{r=1}^{n} (r+1)^3 - r^3$

$= [(n+1)^3 - n^3] + [n^3 - (n-1)^3] \ldots + [3^3 - 2^3] + [2^3 - 1^3]$ M1

$= (n+1)^3 - 1 = n^3 + 3n^2 + 3n$ A1

Also:

$\sum_{r=1}^{n} (r+1)^3 - r^3 = \sum_{r=1}^{n} 3r^2 + 3r + 1$

$= 3\sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$ M1

$= 3\sum_{r=1}^{n} r^2 + \frac{3n(n+1)}{2} + n$ A1A1

Therefore:

$3\sum_{r=1}^{n} r^2 = n^3 + 3n^2 + 3n - \frac{3n(n+1)}{2} - n$ M1

$= n^3 + \frac{3}{2}n^2 + \frac{1}{2}n$

$= \frac{1}{2}n(2n^2 + 3n + 1)$

$= \frac{1}{2}n(n+1)(2n+1)$ A1

Therefore $\sum_{r=1}^{n} r^2 + \frac{n(n+1)(2n+1)}{6}$ AG

[7 marks]

d The coordinate of the bottom right hand corner of the rth rectangle is $\frac{rx}{n}$. M1

The height of the rectangle is $\left(\frac{rx}{n}\right)^2$ A1

So the area of each rectangle is $\frac{x}{n}\left(\frac{rx}{n}\right)^2$ A1

The total area is $\sum_{r=1}^{n} \frac{x}{n}\left(\frac{rx}{n}\right)^2$

Each rectangle has a portion above the curve, so the total area is an overestimate of the true area under the curve. A1

Tip: A diagram would be a great way to form and illustrate this argument!

[4 marks]

e The coordinate of the bottom left hand corner of the rth rectangle is $\frac{(r-1)x}{n}$ M1

The height of the rectangles with top left corner on the curve is $\left(\frac{(r-1)x}{n}\right)^2$ A1

The total area is $\sum_{r=1}^{n} \frac{x}{n}\left(\frac{(r-1)x}{n}\right)^2$

This is less than the area under the curve, so M1

$\frac{x}{n}\sum_{r=1}^{n} \left(\frac{(r-1)x}{n}\right)^2 \leqslant \int_0^x t^2\,dt$ A1

[4 marks]

f $\sum_{r=1}^{n} \frac{x}{n}\left(\frac{rx}{n}\right)^2 = \frac{x^3}{n^3}\sum_{r=1}^{n} r^2 = \frac{x^3}{n^3}\frac{n(n+1)(2n+1)}{6}$ M1

$= x^3 \frac{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6}$ A1

$\frac{x}{n}\sum_{r=1}^{n}\left(\frac{(r-1)x}{n}\right)^2 = \frac{x}{n}\sum_{r=1}^{n-1}\left(\frac{rx}{n}\right)^2$ M1A1

$= \frac{x^3}{n^3}\frac{(n-1)(n)(2n-1)}{6}$

$= x^3 \frac{\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)}{6}$

Taking the limit

$\lim_{n\to\infty} x^3 \frac{\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)}{6} \leqslant \int_0^x t^2\,dt \leqslant \lim_{n\to\infty} x^3 \frac{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6}$ M1

$\frac{x^3}{3} \leqslant \int_0^x x^2\,dx \leqslant \frac{x^3}{3}$

Since $\int_0^x t^2\,dt$ is sandwiched between two quantities tending towards $\frac{x^3}{3}$, it must also tend towards $\frac{x^3}{3}$. A1

[6 marks]

Total [30 marks]

2 a $\bar{X} = \frac{X_1 + X_2}{2}$ A1

[1 mark]

b $E(\bar{X}) = E\left(\frac{1}{2}X_1 + \frac{1}{2}X_2\right) = \frac{1}{2}E(X_1) + \frac{1}{2}E(X_2)$ M1

$= \frac{1}{2}\mu + \frac{1}{2}\mu$ A1

$= \mu$ AG

$\text{Var}(\bar{X}) = \text{Var}\left(\frac{1}{2}X_1 + \frac{1}{2}X_2\right) = \frac{1}{4}\text{Var}(X_1) + \frac{1}{4}\text{Var}(X_2)$ M1

$\frac{1}{4}\sigma^2 + \frac{1}{4}\sigma^2$

$= \frac{1}{2}\sigma^2$ A1

[4 marks]

c i $E(X^2) = \text{Var}(X) + E(X)^2$ A1

ii $E(S^2) = E\left(\frac{X_1^2 + X_2^2}{2} - \bar{X}^2\right) = \frac{1}{2}E(X_1^2) + \frac{1}{2}E(X_2^2) - E(\bar{X}^2)$ M1

$= \frac{1}{2}(\text{Var}(X_1) + E(X_1)^2) + \frac{1}{2}(\text{Var}(X_1) + E(X_1)^2)$ M1

$-(\text{Var}(\bar{X}) + E(\bar{X})^2)$

$= \frac{1}{2}(\sigma^2 + \mu^2) + \frac{1}{2}(\sigma^2 + \mu^2) - \left(\frac{1}{2}\sigma^2 + \mu^2\right)$ A1

$= \frac{1}{2}\sigma^2$ AG

[4 marks]

d i $E(M) = \frac{2}{5}E(X_1) + \frac{3}{5}E(X_2)$ M1

$= \frac{2}{5}\mu + \frac{3}{5}\mu$ A1

$= \mu$ AG

ii $\text{Var}(M) = \frac{4}{25}\text{Var}(X_1) + \frac{9}{25}\text{Var}(X_2)$ M1

$= \frac{13}{25}\sigma^2$ A1

$> \frac{1}{2}\sigma^2$ therefore $\bar{X}$ is a more efficient estimator A1

[5 marks]

e i $L = P(Y = a)P(Y = b)$ M1

$= p(1-p)^{a-1} \times p(1-p)^{b-1}$ A1

	ii	$L = p^2(1-p)^{a+b-2}$	
		$\frac{dL}{dp} = 2p(1-p)^{a+b-2} - (a+b-2)p^2(1-p)^{a+b-3}$	M1A1
		At a max, $\frac{dL}{dp} = 0$	M1
		$p(1-p)^{a+b-3}(2(1-p) - (a+b-2)p) = 0$	M1
		Since $p \neq 0$ and $p \neq 1$ at the maximum value of L	A1
		$2 - 2p = ap + bp - 2p$	
		$2 = ap + bp$	
		$p = \frac{2}{a+b}$	A1
			[8 marks]
f	**i**	$S^2 = \frac{4^2+8^2}{2} - 6^2 = 4$	M1
		Unbiased estimate of $\sigma^2 = 2S^2 = 8$	A1
	ii	$p = \frac{2}{4+8} = \frac{1}{6}$	A1
			[3 marks]
			Total [25 marks]

Practice Set C: Paper 1 Mark scheme

SECTION A

1 **a** Attempt to find x-coordinate of turning point:

$\frac{dy}{dx} = 0 : 4x + 10 = 0$ M1

$x = -\frac{5}{2}$

So required domain: $x \leqslant -\frac{5}{2}$ A1

b $y = 2\left[\left(x + \frac{5}{2}\right)^2 - \frac{25}{4}\right] + 7$ (M1)

$= 2\left(x + \frac{5}{2}\right)^2 - \frac{11}{2}$ A1

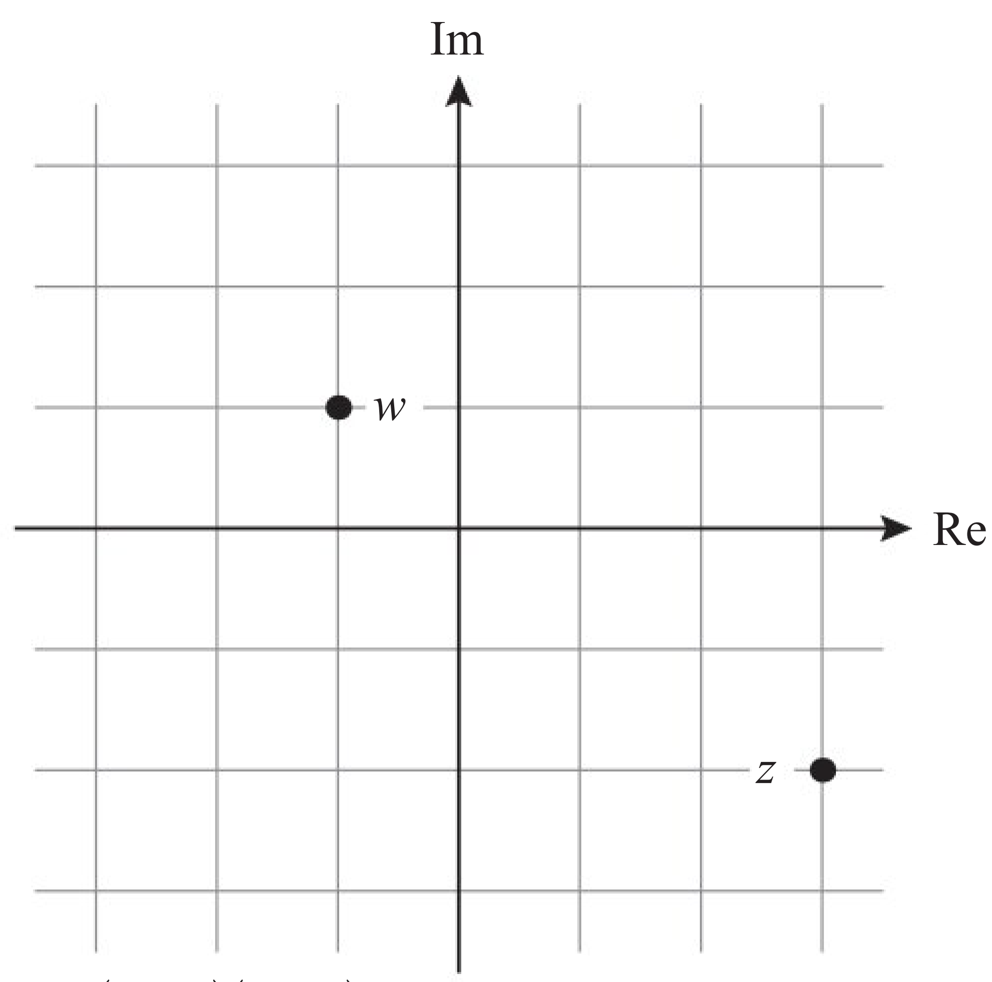
Since $x \leqslant 1$, $f^{-1}(x) = \frac{-5 - \sqrt{2x+11}}{2}$ M1

Domain of $f^{-1} : x \geqslant -\frac{11}{2}$ A1

[6 marks]

2 **a** z correct A1

w correct A1



b $\frac{(-1+i)(3+2i)}{9+4}$ M1

$= -\frac{5}{13} + \frac{1}{13}i$ A1

c Compare real and imaginary parts: M1

$3p - q = 6, -2p + q = 0$

$p = 6, q = 12$ A1

[6 marks]

3 Find the intersection points:

$2x + 1 = x - 3$ OR $2x + 1 = -x + 3$

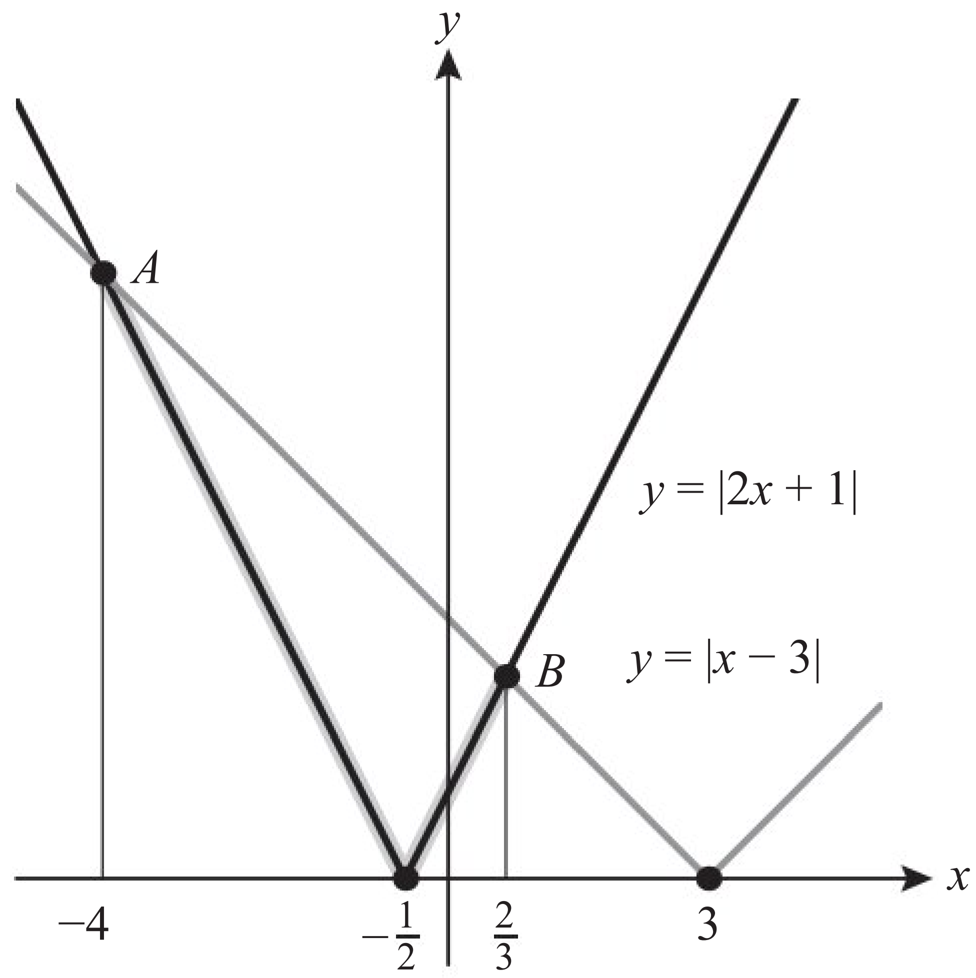
OR

square to get $4x^2 + 4x + 1 = x^2 - 6x + 9$ M1

$x = -4$ A1

$x = \frac{2}{3}$ A1

Graph sketch (or consider signs of factors) M1



$-4 < x < \frac{2}{3}$ A1

[5 marks]

4 To be strictly increasing for all x, f must have no stationary points M1

$f'(x) = 3x^2 + 2kx + k$ A1

$3x^2 + 2kx + k = 0$ has no solutions when $(2k)^2 - 4 \times 3k < 0$ M1

$k(k - 3) < 0$ A1

$0 < k < 3$ A1

[5 marks]

5 Attempt to use partial fractions

$$\frac{3x-16}{(3x-2)(x+4)} = \frac{A}{3x-2} + \frac{B}{x+4}$$

$3x - 16 = A(x + 4) + B(3x - 2)$ M1

$x = -4: -28 = B(-14)$

$B = 2$ A1

$x = \frac{2}{3}: -14 = A\left(\frac{14}{3}\right)$

$A = -3$ A1

$\int_1^6 \frac{2}{x+4} - \frac{3}{3x-2}\,dx = \left[2\ln|x+4| - \ln|3x-2|\right]_1^6$ A1ft

Substitute in limits

$= 2 \ln 10 - \ln 16 - 2 \ln 5 + \ln 1$ M1

$= \ln\frac{1}{4}$ A1

[6 marks]

6 **a** Use $\sin x \approx x$ M1

$\frac{1}{10}\sin 3x \approx \frac{3}{10}x$ A1

b $\frac{3}{10}x \approx x^2$ M1

$x = 0$ A1

$x \approx 0.3$ A1

[5 marks]

7 Use $\frac{u_1}{(1-r)}=5$ M1

Use $u_1 + u_1 r = 3$ M1

Express u_1 from both equations and equate:

$5(1-r)=\frac{3}{1+r}$ M1

$1-r^2=\frac{3}{5}$ A1

$r=\sqrt{\frac{2}{5}}$ A1

[5 marks]

8 EITHER

$\log_4(3-2x)=\frac{\log_{16}(3-2x)}{\log_{16}4}=\frac{\log_{16}(3-2x)}{\frac{1}{2}}$ M1A1

$2\log_{16}(3-2x)=\log_{16}(6x^2-5x+12)$

$\log_{16}(3-2x)^2=\log_{16}(6x^2-5x+12)$ A1

OR

$\log_{16}(6x^2-5x+12)=\frac{\log_4(6x^2-5x+12)}{\log_4 16}=\frac{\log_4(6x^2-5x+12)}{2}$ M1A1

$2\log_4(3-2x)=\log_4(6x^2-5x+12)$

$\log_4(3-2x)^2=\log_4(6x^2-5x+12)$ A1

$(3-2x)^2=6x^2-5x+12$ M1

$2x^2+7x+3=0$ A1

$(2x+1)(x+3)=0$

$x=-\frac{1}{2},-3$ A1

Checks their solutions in equation:

$x=-\frac{1}{2}$: $3-2x=4>0$ and $6x^2-5x+12=16>0$

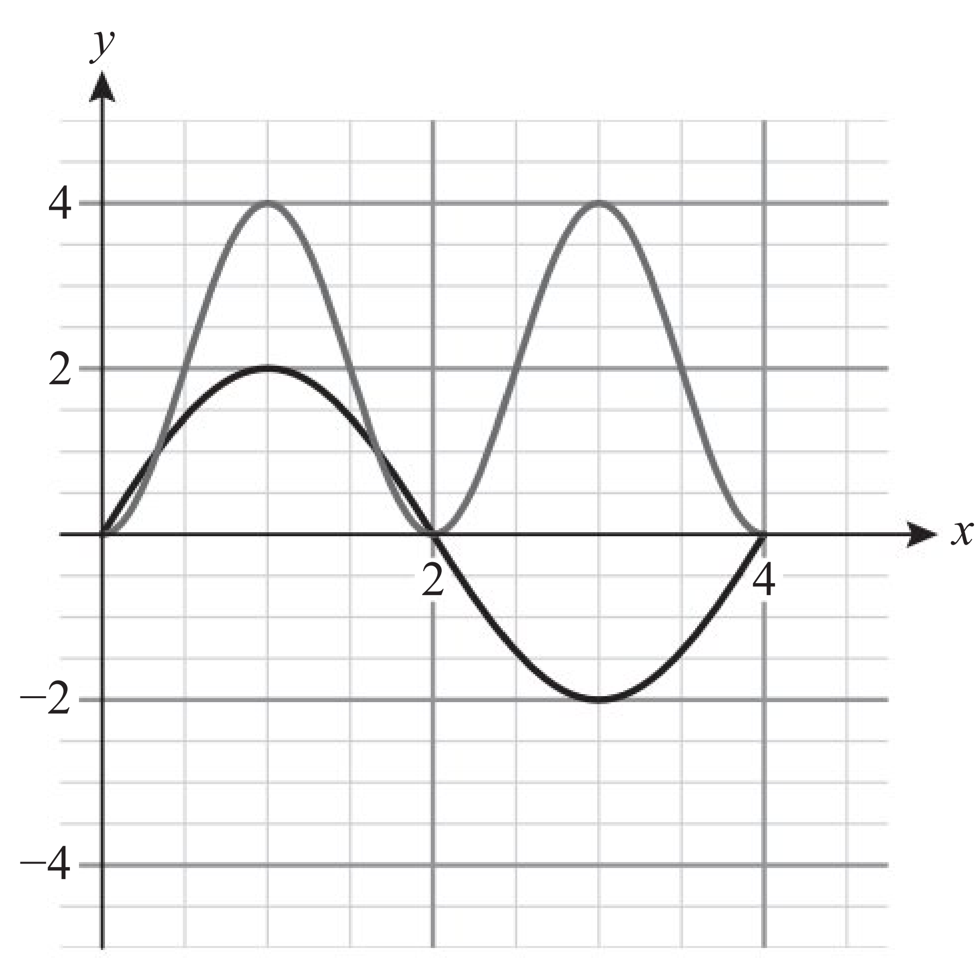
$x=-3$: $3-2x=9>0$ and $6x^2-5x+12=81>0$

So solutions are $x=-\frac{1}{2},-3$

Note: Award A1 if conclusion consistent with working A1

[7 marks]

9 **a**



y in the range 0 to 4 A1

Intersections at $y=0$ A1

Intersections at $y=1$ A1

b Domain: $1 \leqslant x \leqslant 5$ A1

Range: $-4 \leqslant g(x) \leqslant 4$ A1

[5 marks]

10 a $\sin y = x$ (M1)

$\cos\left(\frac{\pi}{2} - y\right) = x$ (M1)

$\arccos x = \frac{\pi}{2} - y$ A1

b $\arcsin x + \arccos x = y + \frac{\pi}{2} - y$ M1

So $\arcsin x + \arccos x \equiv \frac{\pi}{2}$ A1

[5 marks]

SECTION B

11 a i Find $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix}$ A1

$\mathbf{r} = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix}$

OR

$\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix}$ A1A1

ii $\mathbf{r} = \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix}$ A1A1ft

[5 marks]

b i $AB = \sqrt{1^2 + 5^2 + (-4)^2}$ M1

$= \sqrt{42}$ A1

ii $\mathbf{c} = \mathbf{d} \pm 2\overrightarrow{AB}$ (M1)

So the coordinates of C are (1, 13, –5) A1

OR (–3, –7, 11) A1

iii Consider $\overrightarrow{AC_1} \cdot \overrightarrow{AC_2}$ M1

$= 0 - 51 - 64$ $[= -115]$ A1

< 0 so obtuse A1

[8 marks]

c i Use $\overrightarrow{AD} = \begin{pmatrix} -2 \\ 7 \\ 0 \end{pmatrix}$

$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix} \times \begin{pmatrix} -2 \\ 7 \\ 0 \end{pmatrix}$ M1

$= \begin{pmatrix} 28 \\ 8 \\ 17 \end{pmatrix}$ A1

ii Scalar product of $\begin{pmatrix} 28 \\ 8 \\ 17 \end{pmatrix}$ with **a**, **b** or **d** attempted $A(1, -4, 3)$ M1

$28x + 8y + 17z$ M1

$= 47$ A1

[5 marks]

Total [18 marks]

12 a $\cos(2\theta+\theta)=\cos(2\theta)\cos\theta-\sin(2\theta)\sin\theta$ M1

$=(2\cos^2\theta-1)\cos\theta-2\sin^2\theta\cos\theta$ A1

$=2\cos^3\theta-\cos\theta-2(1-\cos^2\theta)\cos\theta$ (M1)

$=4\cos^3\theta-3\cos\theta$ A1

[4 marks]

b i $8\cos^3\theta-6\cos\theta+1=0$ M1

$2(4\cos^3\theta-3\cos\theta)=-1$ (M1)

$\cos 3\theta=-\frac{1}{2}$ A1

ii $3\theta=\frac{2\pi}{3},$ A1

$\frac{4\pi}{3},\frac{8\pi}{3}$ A1

$\theta=\frac{2\pi}{9},\frac{4\pi}{9},\frac{8\pi}{9}$ A1

Hence $x=\cos\left(\frac{2\pi}{9}\right),\cos\left(\frac{4\pi}{9}\right),\cos\left(\frac{8\pi}{9}\right)$ A1

[7 marks]

c Product of the roots of the cubic equation is $-\frac{1}{8}$ M1

$\cos\left(\frac{2\pi}{9}\right)\cos\left(\frac{4\pi}{9}\right)\cos\left(\frac{8\pi}{9}\right)=-\frac{1}{8}$ M1

$8\cos\left(\frac{2\pi}{9}\right)\cos\left(\frac{4\pi}{9}\right)=-\frac{1}{\cos\left(\frac{8\pi}{9}\right)}$

$=-\sec\left(\frac{8\pi}{9}\right)$ A1(AG)

[3 marks]

d State 0 A1

It is the sum of the roots of the equation, the coefficient of x^2 is 0 A1

[2 marks]

Total [16 marks]

13 a $f(-x)=\frac{-x}{1+(-x)^2}$ M1

$=-\frac{x}{1+x^2}$

$=-f(x)$ A1

So f is an odd function A1

[3 marks]

b $\int_0^{\sqrt{3}}\frac{kx}{1+x^2}dx=1$ M1

$\left[\frac{k}{2}\ln(1+x^2)\right]_0^{\sqrt{3}}=1$ A1

$\frac{k}{2}\ln(4)=1$ A1

$k\ln 4^{\frac{1}{2}}=1$ M1

$k=\frac{1}{\ln 2}$ AG

[4 marks]

c $\frac{1}{\ln 2}\int_0^m \frac{x}{1+x^2}\,dx = \frac{1}{2}$ (M1)

$\frac{1}{\ln 2}\frac{1}{2}\ln(1+m^2) = \frac{1}{2}$ A1

$\ln(1+m^2) = \ln 2$ A1

$1 + m^2 = 2$

$m = 1$ A1

[4 marks]

d $g'(x) = \frac{1}{\ln 2}\left(\frac{1(1+x^2) - x(2x)}{(1+x^2)}\right) = 0$ M1A1

$1 - x^2 = 0$

$x = 1$ A1

$g(0) = 0$ and $g(1) = \frac{1}{2\ln 2} > 0$ so $x = 1$ is local maximum (or alternative justification) M1

So $x = 1$ is the mode A1

[5 marks]

e $E(X) = \frac{1}{\ln 2}\int_0^{\sqrt{3}} \frac{x^2}{1+x^2}\,dx$ (M1)

$\frac{x^2}{1+x^2} = \frac{1+x^2-1}{1+x^2} = 1 - \frac{1}{1+x^2}$ M1

$E(X) = \frac{1}{\ln 2}\left[x - \arctan x\right]_0^{\sqrt{3}}$ A1

$= \frac{1}{\ln 2}\left(\sqrt{3} - \arctan\sqrt{3}\right)$ (M1)

$= \frac{1}{\ln 2}\left(\sqrt{3} - \frac{\pi}{3}\right)$ A1

[5 marks]

Total [21 marks]

Practice Set C: Paper 2 Mark scheme

SECTION A

1	**a**	Stratified sampling	A1
	b	Correct regression line attempted	M1
		$y = -1.33x + 6.39$	A1
	c	For every extra hour spent on social media, 1.33 hours less spent on homework.	A1
		No social media gives around 6.39 hours for homework.	A1
			[5 marks]
2		Shaded area $\frac{1}{2}(7.2)^2\,\theta\ (= 25.92\,\theta)$	M1
		Triangle area $\frac{1}{2}(7.2)^2 \sin\theta\ (= 25.92 \sin\theta)$	M1
		$\frac{1}{2}(7.2)^2\,\theta - \frac{1}{2}(7.2)^2 \sin\theta = 9.7$ or equivalent (e.g. $\theta - \sin\theta = 0.3742$)	A1
		Solve their equation using GDC	M1
		$\theta = 1.35$	A1
			[5 marks]
3	**a**	$k + 2k + 3k + 4k = 1$	(M1)
		$k = 0.1$	A1
	b	$\text{E}(X) = k + 4k + 12k + 28k$	(M1)
		$\text{E}(X^2) = k + 8k + 48k + 196k\ (= 25.3)$	M1
		$\text{Var}(X) = 25.3 - [4.5]^2$	(M1)
		$= 5.05$	A1
	c	$25 \times \text{Var}(X)$	(M1)
		$= 126(.25)$	A1
			[8 marks]

4 METHOD 1

Use of $\cot\theta = \frac{1}{\tan\theta}$

$\text{LHS} \equiv \dfrac{\sec\theta\sin\theta}{\tan\theta + \frac{1}{\tan\theta}}$ M1

$\equiv \dfrac{\sec\theta\sin\theta\tan\theta}{\tan^2\theta + 1}$ A1

Use of $\sec^2\theta \equiv \tan^2\theta + 1$

$\equiv \dfrac{\sec\theta\sin\theta\tan\theta}{\sec^2\theta}$ M1

$\equiv \dfrac{\sin\theta\tan\theta}{\sec\theta}$ A1

Express in terms of $\sin\theta$ and $\cos\theta$

$\equiv \sin\theta\dfrac{\sin\theta}{\cos\theta} \times \cos\theta$ M1

$\equiv \sin^2\theta$ AG

[5 marks]

METHOD 2

Use of $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

$\text{LHS} \equiv \dfrac{\sec\theta\sin\theta}{\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}}$ M1

Add fractions in denominator (or multiply through by $\sin\theta\cos\theta$)

$\equiv \dfrac{\sec\theta\sin\theta}{\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}}$ M1

$\equiv \dfrac{\sin^2\theta\sec\theta\cos\theta}{\sin^2\theta + \cos^2\theta}$ A1

$\equiv \dfrac{\sin^2\theta}{\sin^2\theta + \cos^2\theta}$ A1

Use of $\sin^2\theta + \cos^2\theta \equiv 1$

$\equiv \dfrac{\sin^2\theta}{1}$ M1

$\equiv \sin^2\theta$ AG

[5 marks]

5 a Solve $0.003x^3 + 10x + 200 = 720$ using GDC M1
36 cakes A1

b Sketch graph of $y = \dfrac{T(x)}{x}$ M1
Minimum point marked at $x = 32.2$ M1
Minimum = 19.3 minutes A1
Maximum = 21.2 minutes A1

[6 marks]

6 a $20\,C\,6$ (M1)
$= 38\,760$ A1

b Consider two cases: (3 F and 3 NF) or (4 F and 2 NF) M1
$12C3 \times 8C3$ (= 12 320) or $12C4 \times 8C2$ (= 13 860) M1
Both of the above terms seen (not necessarily added for this mark) A1
26 180 selections A1

[6 marks]

7 $\dfrac{du}{dx} = e^x \Rightarrow dx = \dfrac{1}{e^x}du$ M1

$\displaystyle\int \frac{u}{u^2+u-2}\frac{1}{e^x}\,du = \int \frac{1}{u^2+u-2}\,du$ A1

Attempt to use partial fractions

$\dfrac{1}{u^2+u-2} = \dfrac{A}{u-1} + \dfrac{B}{u+2}$ M1

$1 = A(u+2) + B(u-1)$

$A = \dfrac{1}{3}, B = -\dfrac{1}{3}$ A1

$\displaystyle\int \frac{\frac{1}{3}}{u-1} - \frac{\frac{1}{3}}{u+2}\,du = \frac{1}{3}(\ln|u-1| - \ln|u+2|)$ M1

$\ln\left|\dfrac{u-1}{u+2}\right|^{\frac{1}{3}}$ A1

$\displaystyle\int \frac{e^x}{e^{2x}+e^x-2}\,dx = \ln\left|\frac{e^x-1}{e^x+2}\right|^{\frac{1}{3}}(+c)$ A1

[7 marks]

8 Assume there does exist such a function M1

By factor theorem $f\left(-\dfrac{3}{2}\right) = 0$:

$2\left(-\dfrac{3}{2}\right)^3 + b\left(-\dfrac{3}{2}\right)^2 + c\left(-\dfrac{3}{2}\right) + 3 = 0$

Note: award M1 for $f\left(\pm\dfrac{3}{k}\right) = 0$ where $k = 1$ or 2. M1

$3b - 2c - 5 = 0$ A1

By remainder theorem $f(2) = 5$

$2(2)^3 + b(2)^2 + c(2) + 3 = 5$

Note: award M1 for $f(\pm 2) = 5$ M1

$2b + c + 7 = 0$ A1

Solving (1) and (2) simultaneously:

$b = -\dfrac{9}{7}, c = -\dfrac{31}{7}$ A1

This is a contradiction as b, c were assumed to be integers.
So, there exists no such function. A1

[7 marks]

9 $\dfrac{dS}{dt} = 2\pi r\dfrac{dr}{dt}\ldots$ M1

$\ldots + \pi\dfrac{dr}{dt}\sqrt{r^2+25}$ A1

$\ldots + \pi r\dfrac{2r\frac{dr}{dt}}{2\sqrt{r^2+25}}$ M1A1

Substitute $r = 10$, $\dfrac{dr}{dt} = 2$ into their expression M1

$\dfrac{dS}{dt} = 252\,\text{cm}^2\,\text{sec}^{-1}$ A1

[6 marks]

SECTION B

10 a i Arithmetic sequence, $u_1 = 30$, $d = 10$ (M1)
$u_{12} = 30 + 11 \times 10$ M1
$= 140$ A1

ii $S_{12} = 6(60 + 11 \times 10)$ or $\frac{12(30 + 140)}{2}$ M1
$= 1020$ A1

iii $\frac{N}{2}\left(60 + 10(N - 1)\right) = 2000$
OR Create table of values M1
$N = 17.7$
OR $S_{17} = 1870$, $S_{18} = 2070$ A1
In the 18th month A1

[8 marks]

b i Geometric sequence, $u_1 = 30$, $r = 1.1$ (M1)
$S_{12} = \frac{30(1.1^{12} - 1)}{1.1 - 1}$ M1
$= 642$ A1

ii $30 \times 1.1^{N-1} > 100$ M1
$N = 13.6$ (M1)
In the 14th month A1

[6 marks]

c i Multiply answer to **a(ii)** or **b(i)** by the profit at least once M1
Stella: $1020 \times 2.20 = £2244$ A1
Giulio: $642 \times 3.10 = £1990$ A1

ii $\frac{30(1.1^{N} - 1)}{0.1} \times 3.10 > \frac{N}{2}\left(60 + 10(N - 1)\right) \times 2.20$ M1
$N = 22.9$ (M1)
In the 22nd month A1

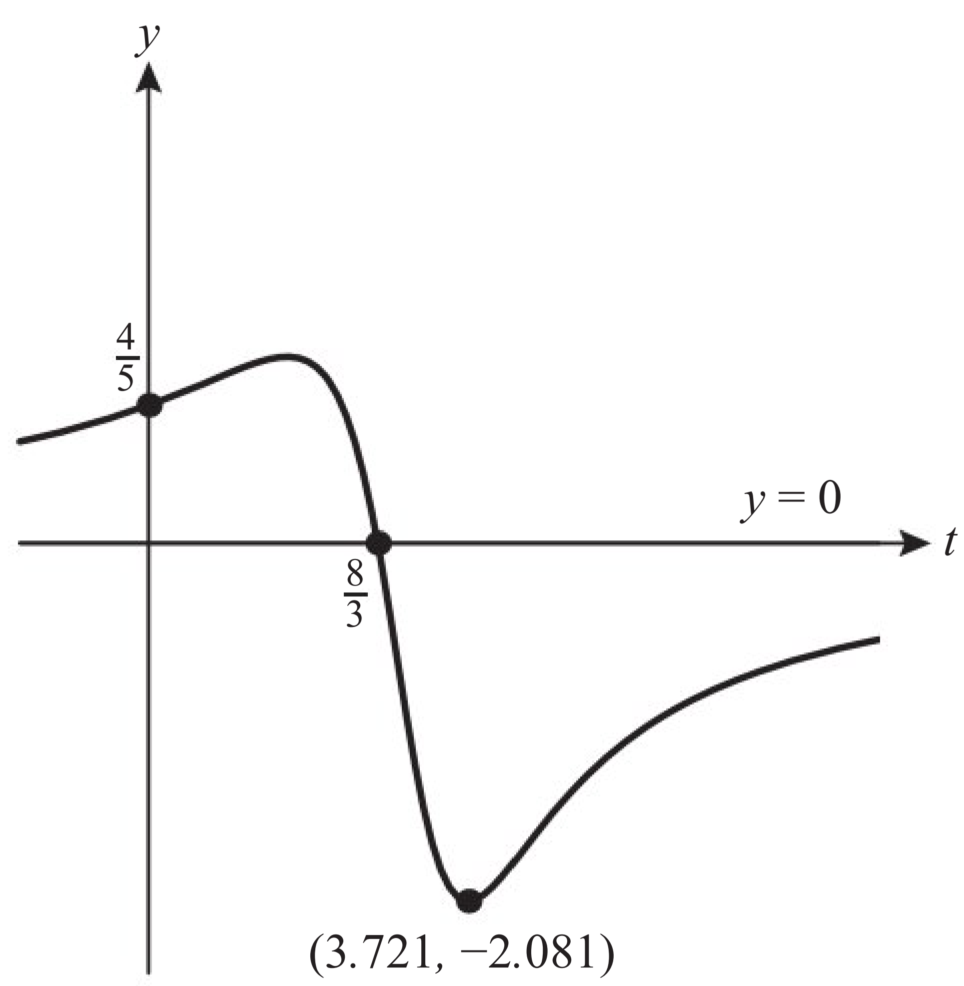
[6 marks]

Total [20 marks]

11 a $v(0) = \frac{8}{10} = 0.8\ \text{m s}^{-1}$ A1

[1 mark]

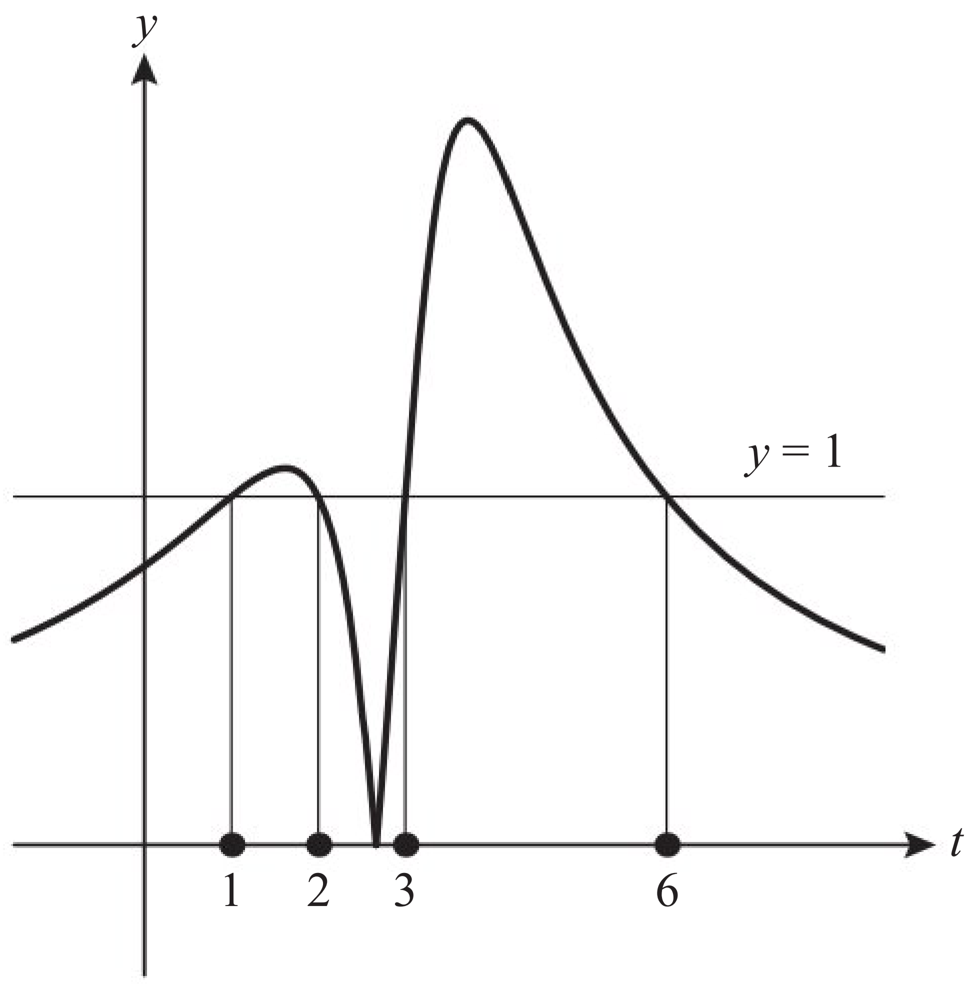
b Sketch graph $y = v(t)$ and identify minimum point. (M1)



Maximum speed = $|-2.08| = 2.08\ \text{m s}^{-1}$
Note: Award M1A0 for $-2.08\ \text{m s}^{-1}$ A1

[2 marks]

c EITHER
$v > 1$ for $1 < t < 2$ M1
$v < 1$ for $3 < t < 6$ M1
OR
Graph $y = |v(t)|$ M1



$|v| > 1$ for $1 < t < 2$ or $3 < t < 6$ M1

So speed > 1 for 4 seconds A1

[3 marks]

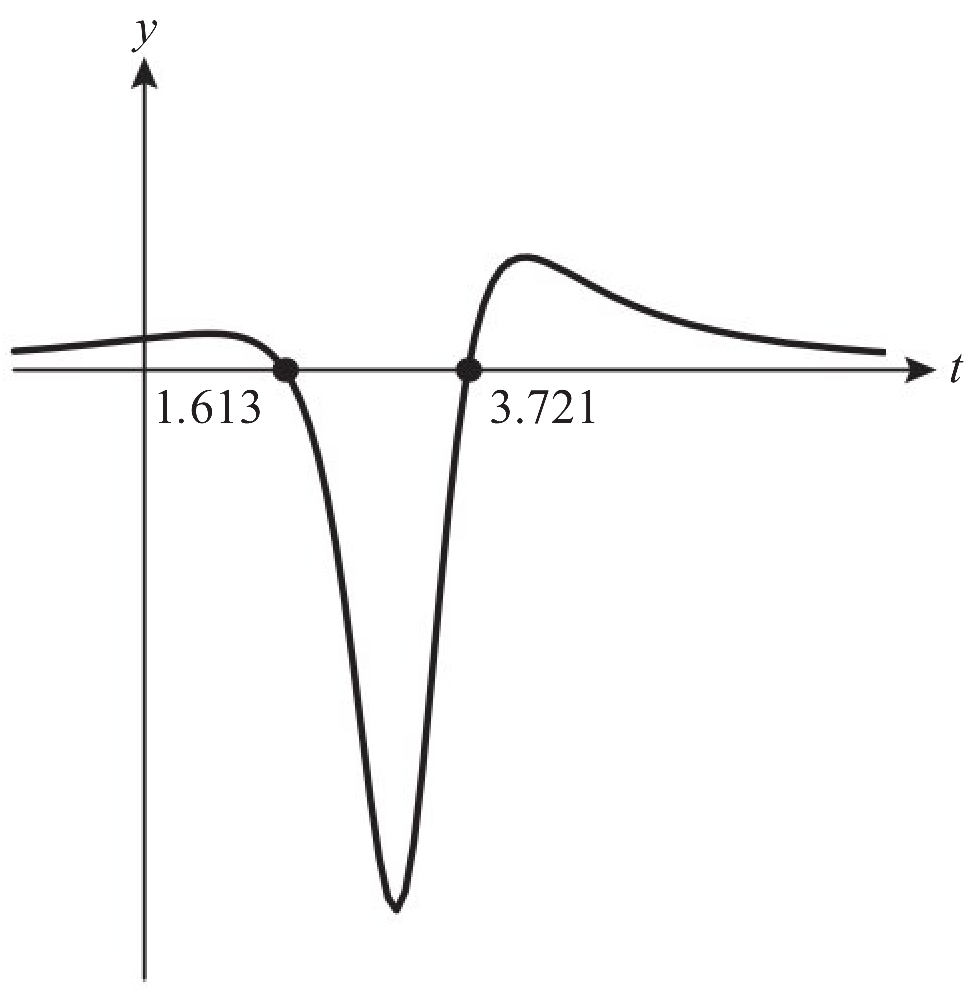
d Object changes direction when $v = 0$ (M1)

$t = \frac{8}{3} = 2.67\,\text{s}$ A1

[2 marks]

e EITHER

Sketch graph of $y = \frac{dv}{dt}$: $y < 0$ for $1.61 < t < 3.72$ (M1)



OR

Use graph of $y = v(t)$: gradient negative for $1.61 < t < 3.72$ (between turning points) (M1)

So $a < 0$ for 2.11 seconds A1

[2 marks]

f From GDC, $\frac{dv}{dt}$ at $t = 5$ … (M1)

… gives $a = 0.52\,\text{m}\,\text{s}^{-2}$ A1

[2 marks]

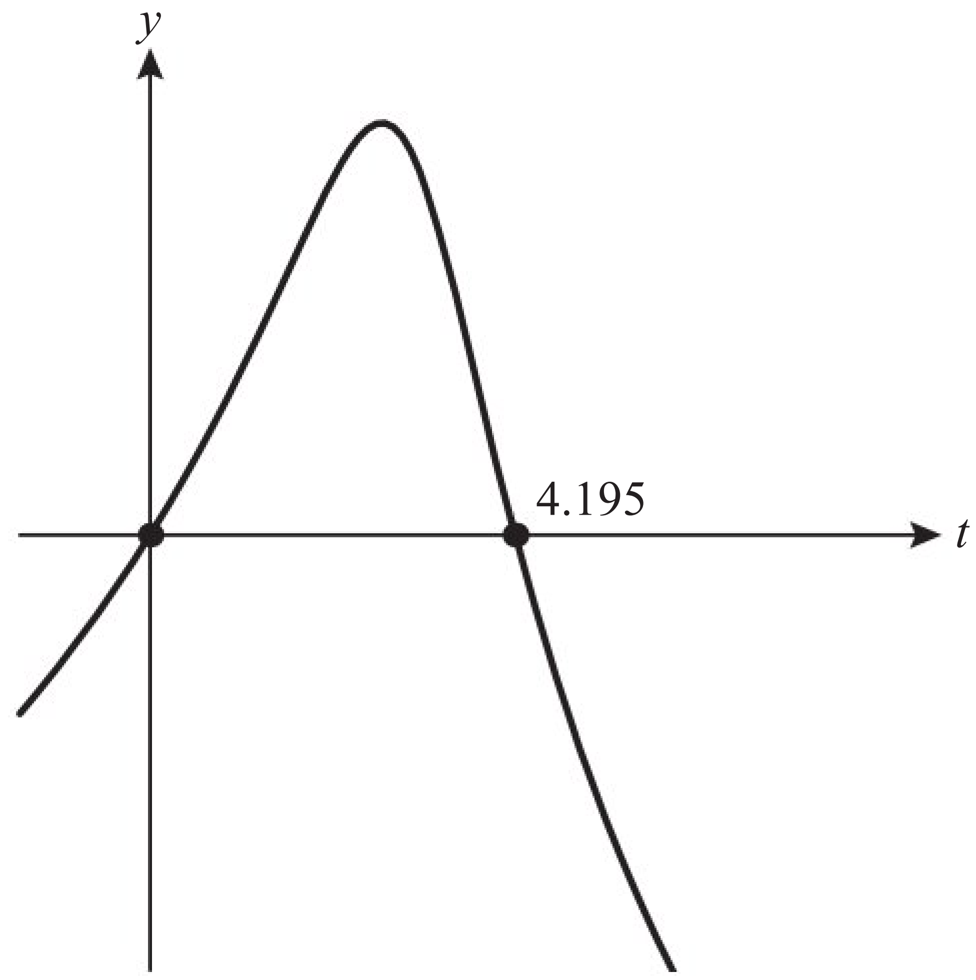
g From GDC:

distance $= \int_0^{10} \left| \frac{8 - 3t}{t^2 - 6t + 10} \right| dt$ M1

$= 9.83\,\text{m}$ A1

[2 marks]

h Sketch graph of $y = \int_0^x v \, dt$ (M1)



Identify x-intercept as being point at which object back at start (M1)

$t = 4.20$ seconds A1

[3 marks]

Total [17 marks]

12 a $\frac{d}{dx}(\ln|\sec x + \tan x|) = \frac{1}{\sec x + \tan x}(\sec x \tan x + \sec^2 x)$ M1A1

$= \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}$ A1

$= \sec x$ AG

[3 marks]

b $\frac{dy}{dx} + \sec x y = \sec x$ M1

Integrating factor:

$e^{\int \sec x \, dx} = e^{\ln|\sec x + \tan x|}$ M1

$= \sec x + \tan x$ A1

$\frac{d}{dx}\left(y(\sec x + \tan x)\right) = \sec^2 x + \sec x \tan x$ M1A1

$y(\sec x + \tan x) = \int \sec^2 x + \sec x \tan x \, dx$

$y(\sec x + \tan x) = \tan x + \sec x + c$ A1

$y = 1 + \frac{c}{\sec x + \tan x}$ A1

[7 marks]

c i $\frac{d^3y}{dx^3} - \sin x \frac{dy}{dx} + \cos x \frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$ M1A1

$\frac{d^3y}{dx^3} = (\sin x - 1)\frac{dy}{dx} - \cos x \frac{d^2y}{dx^2}$ AG

ii Substitute given values into differential equation:

When $x = 0$

$\frac{d^2y}{dx^2} + \cos 0(1) + 2 = 1$ M1

$\frac{d^2y}{dx^2} = -2$ A1

Substitute their value into expression for $\frac{d^3y}{dx^3}$:

When $x = 0$

$\frac{d^3y}{dx^3} = (\sin 0 - 1)(1) - \cos 0(-2)$ M1

$= 1$ A1

Substitute their values into Maclaurin series

$y = 2 + x - \frac{2}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$ M1

$2 + x - x^2 + \frac{1}{6}x^3 + \cdots$ A1

[8 marks]

Total [18 marks]

Practice Set C: Paper 3 Mark scheme

1 **a** ()() (()) (()(A2

)()( ))(( ))()

[2 marks]

b **i** 16 A1

ii 8 A1

iii 12 870 A1

${}^{2n}C_n$ A1

[4 marks]

c **i** ()() (()) A1

ii ((())) ()(()) ()()() (())() (()()) M1

So $B_3 = 5$ A1

[3 marks]

d $\frac{B_1}{A_1} = \frac{1}{2}$ $\frac{B_2}{A_2} = \frac{1}{3}$ $\frac{B_3}{A_3} = \frac{5}{20} = \frac{1}{4}$ $\frac{B_8}{A_8} = \frac{1}{9}$ (M2)

This suggests $f(n) = \frac{1}{n+1}$ A1

$B_n = \frac{1}{n+1}\,{}^{2n}C_n$

[3 marks]

e When $n = 1$ M1

$B_1 = \frac{1}{2} \times {}^2C_1 = \frac{1}{2} \times 2 = 1$

So the conjecture is true when $n = 1$ A1

Assume that it is true when $n = k$ M1

$B_1 = \frac{1}{k+1}\,{}^{2k}C_k = \frac{1}{k+1}\,\frac{(2k)!}{k!k!}$ A1

Then using the given recursion relation:

$B_{k+1} = \frac{4k+2}{k+2} \times \frac{1}{k+1}\,\frac{(2k)!}{k!k!}$ M1

$= \frac{2(2k+1)}{(k+2)(k+1)} \times \frac{(2k)!}{k!k!}$

$= \frac{2(2k+1)}{(k+2)(k+1)} \times \frac{(2k)!}{k!k!} \times \frac{(2k+1)(2k+2)}{(2k+1)(2k+2)}$ M1

$= \frac{(2k+1)}{(k+2)(k+1)} \times \frac{(2k)!}{k!k!} \times \frac{(2k+1)(2k+2)}{(2k+1)(k+1)}$

$= \frac{1}{k+2} \times \frac{2(k+1)!}{(k+1)!(k+1)!}$ A1

$= \frac{1}{(k+1)+1}\,{}^{2(k+1)}C_{k+1}$

So if the statement works for $n = k$ then it works for $n = k + 1$ and it works for $n = 1$ therefore it works for all $n \in \mathbb{Z}^+$ A1

[8 marks]

Tip: You might wonder where the given recursion relation comes from. The most natural way is from the triangulation of a polygon interpretation of Catalan numbers.

f $\frac{B_{20}}{A_{20}} = f(20) = \frac{1}{21}$ M1A1

[2 marks]

g Let (be equivalent to a vote for Elsa and) be equivalent to a vote for Asher M1

Then the total number of ways of ending in a draw is A_{50} and the number where Asher is never ahead is B_{50} M1

The probability is then $\frac{B_{50}}{A_{50}} = \frac{1}{51}$ A1

[3 marks]

Total [25 marks]

2 **a** $\left|e^{\frac{2\pi i}{3}} - 1\right| = \left|\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} - 1\right|$ M1

$= \left|-\frac{1}{2} + \frac{\sqrt{3}}{2}i - 1\right|$ A1

$= \sqrt{\left(-\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$= \sqrt{3}$ A1

[3 marks]

b Bill: $e^{\frac{2\pi i}{3}}$ A1

Charlotte: $e^{\frac{4\pi i}{3}}$ A1

[2 marks]

c Using part **a**: $\sqrt{3}$ units in $\sqrt{3}$ seconds A1A1

[2 marks]

d The direction from z_A to z_B is $z_B - z_A$ A1

The distance travelled per unit time is one, so this is $\frac{z_B - z_A}{|z_B - z_A|}$ A1

[2 marks]

e $z_B = e^{\frac{2\pi i}{3}} z_A$ A1

[1 mark]

f $\frac{dz_A}{dt} = \frac{dr}{dt}e^{i\theta} + ire^{i\theta}\frac{d\theta}{dt}$ M1A1

[2 marks]

g $\frac{dr}{dt}e^{i\theta} + ire^{i\theta}\frac{d\theta}{dt} = \frac{e^{\frac{2\pi i}{3}} z_A - z_A}{\left|e^{\frac{2\pi i}{3}} z_A - z_A\right|} = \frac{z_A\left(e^{\frac{2\pi i}{3}} - 1\right)}{|z_A|\left|e^{\frac{2\pi i}{3}} - 1\right|}$ M1A1

$= \frac{re^{i\theta}\left(e^{\frac{2\pi i}{3}} - 1\right)}{r\left|e^{\frac{2\pi i}{3}} - 1\right|}$ M1A1

$= \frac{e^{i\theta}}{\sqrt{3}}\left(-\frac{3}{2} + \frac{i\sqrt{3}}{2}\right)$ A1

$= e^{i\theta}\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

Dividing through by $e^{i\theta}$:

1. $\frac{dr}{dt} + ir\frac{d\theta}{dt} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$

Comparing real and imaginary parts:

$\frac{dr}{dt} = -\frac{\sqrt{3}}{2}$ A1

$r\frac{d\theta}{dt} = \frac{1}{2}$ A1

[7 marks]

h $r = -\frac{\sqrt{3}}{2}t + c$ M1

When $t = 0$, $r = 1$ so $c = 1$ M1

$r = 1 - \frac{\sqrt{3}}{2}t$ A1

$\frac{d\theta}{dt} = \frac{1}{2\left(1 - \frac{\sqrt{3}}{2}t\right)} = \frac{1}{2 - \sqrt{3}t}$ M1

$\theta = -\frac{1}{\sqrt{3}}\ln(2 - \sqrt{3}t) + c$ A1

When $t = 0$, $\theta = 0$ so $c = \frac{1}{\sqrt{3}}\ln 2$ M1

$\theta = \frac{1}{\sqrt{3}}\ln\left(\frac{2}{2 - \sqrt{3}t}\right)$ A1

[7 marks]

i Meet when $r = 0$ M1

This happens when $1 - \frac{\sqrt{3}}{2}t = 0$

So $t = \frac{2}{\sqrt{3}}$ A1

Since $v = 1$ the distance travelled is $\frac{2}{\sqrt{3}}$ units A1

As $t \to \frac{2}{\sqrt{3}}$, $\theta \to \infty$ so the snails make an infinite number of rotations A1

[4 marks]

Total [30 marks]

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